Boundary Regularity for Conformally Compact Einstein Metrics in Even Dimensions

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Abstract

We study boundary regularity for conformally compact Einstein metrics in even dimensions by generalizing the ideas of Michael Anderson found in [And03] and [And06]. Our method of approach is to view the vanishing of the Ambient Obstruction tensor as an nth order system of equations for the components of a compactification of the given metric. This, together with boundary conditions that the compactification is shown to satisfy provide enough information to apply classical boundary regularity results. These results then provide local and global versions of finite boundary regularity for the components of the compactification.

1 Introduction

In recent years, both the mathematics and theoretical physics communities have shown a great deal of interest in studying the analysis and geometry of conformally compact Einstein metrics. In particular, much progress has been made recently with regard to boundary regularity of these metrics. This paper provides one approach for investigating this topic and studies the problem of regularity in the even dimensional case.

The concept of a conformally compact metric was developed in 1963 by Roger Penrose [PR88]. Since then, such metrics have been studied in [LeB82, MM87, Maz88, GL91] along with many others; see [Lee06] and the references therein, for example. The physics community has also become interested in conformally compact Einstein metrics since the introduction of the so called AdS/CFT correspondence by Maldacena [Mal98]. See for example [Wit98, HS98, dHSS01] along with the references therein.

An example stemming from the developments in physics, which shows how the issue of boundary regularity of a conformally compact Einstein metric arises, is the volume renormalization of a conformally compact Einstein manifold. This renormalization gives rise to an invariant associated with the conformal boundary manifold [Gra00]. In order to obtain the result, there must be some compactification for the Einstein metric which is smooth enough at the boundary for certain calculations to be made.

The first result concerning boundary regularity of conformally compact Einstein metrics was negative. Namely, Fefferman and Graham [FG85] showed that if M is odd dimensional, there are examples where there is no smooth compactification despite smoothness of the boundary metric. More recently, though, positive results have been proved. In [And03], Anderson studies the four dimensional case by considering the Bach tensor of a compactification $g = \rho^2 g_+$ with constant scalar curvature. The Bach tensor is a classically known natural tensor depending on two derivatives of curvature which is conformally invariant in dimension four, and vanishes for Einstein metrics. Making use of these facts and working in special harmonic coordinates for g, Anderson generates second order uniformly elliptic equations for components of the Ricci tensor of g, and for the components of g itself. He also derives boundary equations for this system. A bootstrap argument is then applied to prove boundary regularity. In [And06], he presents a revised version of the argument based on viewing the Bach equation, combined with his boundary equations, as a fourth order elliptic boundary value problem in the sense of [ADN64, Mor66] for the components of the constant scalar curvature compactification.

In another recent paper, Chruściel, Delay, Lee, and Skinner [CDLS05] study boundary regularity in general dimension by applying the uniformly degenerate theory of [AC96] directly to a gauge broken

Einstein equation. They show that in the case when the boundary metric is C^∞, there is a compactification which, in suitable coordinates, is C^∞ up to the boundary in even dimensions, and which has an asymptotic expansion involving logarithms in odd dimensions.

In this paper, we follow Anderson's first approach to prove finite boundary regularity in general even dimensions. The Bach tensor is not conformally invariant in dimensions other than four, but there is a generalization of the Bach tensor in each higher even dimension called the ambient obstruction tensor [FG85, GH05]. Like the Bach tensor, it is conformally invariant and vanishes for Einstein metrics. We choose a constant scalar curvature compactification for the given Einstein metric and we work in harmonic coordinates. In this setting, the vanishing of the ambient obstruction tensor will provide us with a system of equations that are uniformly elliptic up to the boundary. However, these equations will be nth order and this brings up the difficulty of finding the necessary boundary equations to make a well posed boundary value problem. The natural boundary data is that of a boundary metric, and nothing more. A large task, then, is to derive new boundary equations that the compactification satisfies and that make a well posed problem. Once such a boundary value problem is derived, classical theorems may be applied to yield boundary regularity results.

We focus on metrics that satisfy the specific Einstein equation $Ric_+ = -(n-1)g_+$ and henceforth, unless mentioned otherwise, this is always what we mean when we say a metric is Einstein. We say that g_+ is $C^{m,\sigma}$ conformally compact if for any $C^{m+1,\sigma}$ defining function, the resulting compactified metric $g = \rho^2 g_+$ is $C^{m,\sigma}$ up to the boundary (which makes sense as long as $\overline{M}$ has a $C^{m+1,\sigma}$ structure). The local and global versions of our main result are as follows:

Theorem A. *Let $\overline{M}$ be an n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. Let $p \in \partial M$ and let $U \subset \overline{M}$ be a neighborhood of p with boundary portion $D = U \cap \partial M$. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on $U \cap M$. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(D)$, where $k \geq r$ and $0 < \gamma < 1$. Given a C^∞ coordinate system on the boundary in a neighborhood of p, there is a coordinate system on a neighborhood $V \subset U$ of p that is $C^{r,\sigma}$ compatible with the given smooth structure and which restricts to the given coordinates on the boundary, and there is a defining function $\rho \in C^{r-1,\sigma}(V)$ in the new coordinates, such that $\rho^2 g_+$ has boundary metric h and in the new coordinates, $\rho^2 g_+ \in C^{k,\gamma}(V)$.*

Theorem B. *Let $\overline{M}$ be a compact n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on M. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(\partial M)$, where $k \geq r$ and $0 < \gamma < 1$. Then there is a $C^{r,\sigma}$ diffeomorphism $\Psi : \overline{M} \longrightarrow \overline{M}$ which restricts to the identity on the boundary and there is a defining function $\check{\rho} \in C^{r-1,\sigma}(\overline{M})$ such that $\check{\rho}^2 \Psi^*(g_+)$ has boundary metric h and $\check{\rho}^2 \Psi^*(g_+) \in C^{k,\gamma}(\overline{M})$.*

We note that the coordinates in Theorem A depend on g_+ and h, but not their regularity. Hence, it follows that if h is C^∞, then the new coordinates are C^∞, and $\rho^2 g_+$ is C^∞ in these new coordinates, for some defining function $\rho \in C^{r-1,\sigma}(\overline{M})$. Similarly, if h is C^∞ in Theorem B, then $\check{\rho}^2 \Psi^*(g_+)$ is C^∞ for some $\check{\rho} \in C^{r-1,\sigma}(\overline{M})$.

Anderson argues in [And03] and [And06] that a version of these results holds for $n = 4$ when a conformal compactification of g_+ is $L^{2,p}$ for some $p > 4$, instead of $C^{r-1,\sigma}$, $r \geq 4$. I am unable to verify the theorem with this hypothesis. Also, Anderson's statement does not mention the change of coordinates, nor the fact that defining function may not be smooth in the new coordinates.

Observe that in Theorem A we cannot conclude that g_+ is $C^{k,\gamma}$ conformally compact in the new coordinates, since the defining function need not be $C^{k+1,\gamma}$ in the new coordinates. Similarly, we cannot conclude that $\Psi^*(g_+)$ is $C^{k,\gamma}$ conformally compact in Theorem B. This is a consequence of our specific compactification as well as the change of coordinates. I expect that g_+ is $C^{k,\gamma}$ conformally compact in the new coordinates. This can be reduced to a regularity question for the singular Yamabe problem, since as an Einstein metric, g_+ is a (singular) constant scalar curvature metric in its conformal class. Using results of [AC96] on this problem, we are able to conclude the following theorem:

Theorem C. *Let $\overline{M}$ be a compact n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on M. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(\partial M)$, where $k \geq r$ and $k \geq n+1$, and $0 < \gamma < 1$. Then there is a $C^{r,\sigma}$ diffeomorphism $\Psi : \overline{M} \longrightarrow \overline{M}$ which restricts to the identity on ∂M such that $\Psi^*(g_+)$ is $C^{k,\gamma'}$ conformally compact for some γ', $0 < \gamma' \leq \gamma$.*

Similarly to the situation with Theorems A and B, this result allows us to conclude that $\Psi^*(g_+)$ is C^∞ conformally compact in the new smooth structure if h is C^∞.

Theorem C is deficient in that, when $r = n$, it should also hold with $k = n$ and in general, it should hold with $\gamma' = \gamma$, and there should be an analogous local result. These deficiencies are a direct consequence of corresponding deficiencies in the regularity theorem of [AC96]. Hence, improvement of their theorem would result in improvement of Theorem C.

The regularity theorem of [CDLS05] using uniformly degenerate methods assumes that g_+ is defined in a collar neighborhood of the full boundary, g_+ is C^2 conformally compact, and $h \in C^\infty(\partial M)$. Our use of high order elliptic equations and boundary conditions has the disadvantage that it seems to necessitate the high a priori regularity assumptions in Theorems A–C. On the other hand, Theorem A is local, and all three theorems apply for boundary metrics of finite regularity.

The outline of this paper is as follows. We begin, in Section 2, by developing the mathematical background for studying conformally compact Einstein metrics. We define precisely what a conformally compact metric is and introduce a number of related notions. We also discuss boundary adapted harmonic coordinates and discuss their properties. To finish the section, we introduce the constant scalar curvature compactification.

In Section 3, we recall the notions of geodesic compactification and geodesic coordinates. The geodesic compactification for a conformally compact Einstein metric in its geodesic coordinates has a particularly simple form. This then can be used to express curvature tensors of the compactification at the boundary purely in terms of the boundary metric. Unfortunately, working with this compactification in these coordinates introduces a loss of regularity. To avoid this loss, we construct an approximate version of the geodesic compactification and use the approximate geodesic coordinates of [AC96] for this compactification. We then show that at the boundary, these "almost geodesic compactifications" produce the same expressions as exact geodesic compactifications for derivatives up to a fixed finite order of curvature tensors in terms of the boundary metric.

Section 4 is dedicated to the derivation of the boundary value problem. The equations on the interior will come from expressing the ambient obstruction tensor for a constant scalar curvature compactification in harmonic coordinates. In deriving boundary conditions, we will find that some of the equations come to us naturally, while others require a careful analysis of curvature tensors at the boundary. The results for almost geodesic compactifications will play a major role in this analysis. Our first order boundary conditions are different from and simpler than those of Anderson. To derive them, we will not need to require that our harmonic coordinates restrict to harmonic coordinates for h on the boundary.

Theorems A and B are proved in Section 5. To prove Theorem A, we treat the nth order boundary value problem as a sequence of second order problems much like Anderson's approach in [And03]. An analysis of the boundary equations shows that they are sufficient for the application of boundary regularity results to these second order equations. Following this, an involved bootstrap yields the result. With Theorem A proved, Theorem B then follows by a patching argument and an approximation theorem.

Section 6 contains our discussion of regularity of the defining function. The regularity theorem of [AC96] asserts the existence of an asymptotic expansion involving log terms for the solution of the singular Yamabe problem. We show that in the case that the Yamabe metric is actually Einstein, then no log terms occur. This along with Theorem B is used to prove Theorem C.

Finally, section 7 discusses an alternative approach to the proof of Theorem A, which is to treat the problem as a single elliptic boundary value problem. This is similar to Anderson's analysis in [And06].

I would like to thank Robin Graham for his help on this paper. The many discussions we had on these topics were indispensable and without his patience and guidance, this paper would not exist.

2 Background

To begin, we set up the situation in which we will be working primarily. In doing so, principal definitions and notions are introduced, along with a number of facts and tools that will be used later. We also introduce our notation and conventions that will be followed throughout this paper.

2.1 Preliminaries and Main Definitions

For the most part, we will be working on or near the boundary of an n-dimensional C^∞ manifold with boundary $\overline{M}$, with interior M, and C^∞ boundary ∂M. When working locally, let U be an open set in $\overline{M}$ containing a nonempty boundary portion $D = U \cap \partial M$. In general, the exact choice of U is not crucial, and so as necessary, and without explicit mention, U may be shrunk to a smaller domain, still containing part of the boundary, so that all relevant notions are well defined. A function ρ on U is a defining function for D if $\rho|_D = 0$, $d\rho|_D \neq 0$, and for convention, ρ is positive in M.

Let $m \in \mathbb{N} \cup \{0\}$, and $0 < \sigma < 1$. We say a metric g_+ on $U \cap M$ is $C^{m,\sigma}$ (resp. C^m, C^∞) *conformally compact* if $\rho^2 g_+$ extends to a $C^{m,\sigma}$ (resp. C^m, C^∞) metric on U, where ρ is a $C^{m+1,\sigma}$ (resp. C^{m+1}, C^∞) defining function for D. Note that this makes sense as long as $\overline{M}$ has a $C^{m+1,\sigma}$ (resp. C^{m+1}, C^∞) structure. For a given ρ, let $g = \rho^2 g_+$ be the extended metric. Then we say g is a *compactification* for g_+. Letting $\iota : D \to U$ be inclusion, we call $h = \iota^* g$ a *boundary metric*. The equivalence class $[h]$ whose elements arise from various choices of ρ is called the *conformal infinity* of g_+.

It follows from these definitions that for any conformally compact metric g_+, the restriction to ∂M of $|d\rho|_{\rho^2 g_+}$ is invariant with respect to the choice of ρ. If this invariant is constantly equal to 1 and $m \geq 2$, then the sectional curvatures of g_+ all approach -1 as we approach the boundary. See [And03] or [Maz88] for details. We say that g_+ is *asymptotically hyperbolic* on U if g_+ is conformally compact and $|d\rho|^2_{\rho^2 g_+} = 1$ on D. If g_+ is C^m conformally compact on U and $\rho^{-1}(1 - |d\rho|^2_{\rho^2 g_+}) \in C^m(U)$, then we say g_+ is C^m *asymptotically hyperbolic* on U. It is straightforward to check that this definition is independent of the choice of C^{m+2} defining function. Also observe that if g_+ is asymptotically hyperbolic and C^{m+1} conformally compact, then it is C^m asymptotically hyperbolic.

We study conformally compact Einstein metrics, by which we mean conformally compact metrics g_+ satisfying the Einstein equation $Ric_+ = -(n-1)g_+$. Such metrics are asymptotically hyperbolic, but we will sometimes need to use the stronger condition mentioned above. An Einstein metric that is $C^{m,\sigma}$ conformally compact will be called a $C^{m,\sigma}$ *conformally compact Einstein metric*. Similarly, an Einstein metric which is C^m asymptotically hyperbolic will be called a C^m *asymptotically hyperbolic Einstein metric*.

Because we will be working with various metrics, objects associated to a given metric will be adorned with the same symbol. For example, the Ricci curvature tensor associated to g_+ will be denoted Ric_+. If a symbol is unadorned, then it corresponds to the unadorned metric g. Also, we note here that we do not change notation to distinguish between various representations of a metric in different coordinates. As this has the potential to generate confusion, we make clear what is happening whenever we are dealing with multiple coordinate systems at once.

2.2 Boundary Adapted Harmonic Coordinates

A coordinate system is called a *boundary adapted coordinate system* if it has the property that one of the coordinate functions is a defining function for the boundary. Note that in such a coordinate system, the remaining coordinates restrict to a coordinate system on ∂M.

We will be working with such coordinate systems throughout, so we introduce some conventions. Let $\{x^\alpha\}$ be a boundary adapted coordinate system. Then, x^0 will always be a defining function, while x^i ($i \neq 0$) will always refer to the remaining coordinates. Consistent with this, any use of indices will follow the convention that Roman indices will range from 1 to $n-1$, while Greek indices range from 0 to $n-1$.

While working with tensors in coordinates and to represent terms of order less than the principal part that appear in various equations, we will often have a need to encapsulate terms depending on a certain number of derivatives of the metric or other tensors. With this in mind, let $\mathcal{P}(\partial^{p_1} s_1, \partial^{p_2} s_2, \ldots)$ denote an expression whose components are polynomials in the components of the tensors s_r and their coordinate derivatives up to order p_r. We will use "∂_t" to represent derivatives with respect to coordinates x^i, while "∂_0" indicates differentiation only with respect to x^0. Also, the use of "$\nabla^p s$" instead of "$\partial^p s$" will indicate dependence on the components of covariant derivatives of s up to order p. Finally, different instances of this notation will not mean the same expression.

When there is no chance for confusion, "∂_α" or a comma followed by "α" will be used to denote coordinate differentiation with respect to the coordinate function x^α. Covariant derivatives will be represented with a "∇_α".

For later reference, we note that the Riemannian Laplacian of a function f with respect to a metric g is given by $\Delta f = \mathrm{tr}_g(\nabla^2 f)$ so that in coordinates, $\Delta f = g^{\alpha\beta}\partial_\alpha\partial_\beta f + \mathcal{P}(g^{-1}, \partial g, \partial f)$. More generally, letting $\mathcal{L}_l f$ be the l-fold trace of $2l$ coordinate derivatives of f, we have $\Delta^l f = \mathcal{L}_l f + \mathcal{P}(g^{-1}, \partial^{2l-1} g, \partial^{2l-1} f)$.

A coordinate system on $\overline{M}$ is called a *harmonic coordinate system* for a metric g if each of the coordinate functions is harmonic with respect to g. It will turn out that boundary adapted harmonic coordinates will prove most beneficial when it comes to questions of regularity. There are two main reasons for this. First, the regularity of a metric is preserved when transforming to its harmonic coordinates. Second, the components of the Ricci tensor can be expressed as $\mathcal{L}$ acting on components of the metric plus lower order terms. These facts were established and existence of harmonic coordinates about a point was proved in [DK81]. To account for a boundary, we have the following proposition.

Proposition 2.1. *Let $U \subset \overline{M}$ be a coordinate domain with boundary portion D and let $\{x^i\}$ be coordinates for D which are compatible with the coordinates on U. Let g be a metric in $C^{m,\sigma}(U)$, where $m \geq 1$ and $0 < \sigma < 1$. Then near any point $p \in D$, there exists a boundary adapted harmonic coordinate system for g that is $C^{m+1,\sigma}$ related to the given coordinates on U and that restricts to the given coordinates $\{x^i\}$ on the boundary.*

Proof. Let $\{y^\alpha\}$ be a C^∞ boundary adapted coordinate system in a neighborhood of p such that when $y^0 = 0$, the remaining coordinates $\{y^i\}$ restrict to $\{x^i\}$. Consider an open set V with the following properties. First, V is the interior of a C^∞ manifold with boundary, diffeomorphic to an open n-dimensional ball. Second, $\partial V \cap \partial M$ is diffeomorphic to a closed $(n-1)$-dimensional ball in ∂M, and contains p in its interior. Third, $\overline{V}$ is in the domain of the coordinate system $\{y^\alpha\}$. Now, construct harmonic functions x^α near p by solving the Dirichlet problem

$$\begin{cases} \Delta x^\alpha = 0 \\ x^\alpha|_{\partial V} = y^\alpha \end{cases}$$

Existence of such functions can be found in [GT01]. Moreover, by the maximum principle, we know $x^0 > 0$ in V, and by Hopf's Lemma, we know that $dx^0|_{\partial V \cap \partial M} \neq 0$. Hence, x^0 is a defining function for ∂M near p.

The set of functions $\{x^0, x^i; i = 1, \ldots, n-1\}$ will form coordinates on an open set W near p. By elliptic boundary regularity (see [GT01]), these coordinate functions are $C^{m+1,\sigma}(W)$ with respect to $\{y^\alpha\}$, since g is in $C^{m,\sigma}(W)$. $\square$

Observe that g is $C^{m,\sigma}$ in these new coordinates since viewing g after a coordinate change involves only first derivatives of the new coordinates. Details related to this can be found in [DK81]. Moreover, since the construction here does not alter the boundary coordinates, the regularity of h is preserved as well.

Besides preserving the regularity of g, harmonic coordinates also produce a simplified formula for the Ricci tensor, which will turn out to have a number of applications in this paper.

Lemma 2.2. *In harmonic coordinates for g,*

$$Ric_{\alpha\beta} = -\frac{1}{2}\mathcal{L} g_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial g). \tag{1}$$

This lemma and its proof can be found in [DK81]. It is basically a computation and we indicate the key ingredients here. First, we note that having a harmonic coordinate system with respect to g is equivalent to having traces of the Christoffel symbols all equal to zero:

$$\Gamma^\alpha = g^{\beta\delta}\Gamma^\alpha_{\beta\delta} = 0.$$

We can differentiate this equation and use the resulting conditions to reduce a standard formula for Ricci to (1).

2.3 Constant Scalar Curvature Compactification

As indicated in the introduction, we will find that making an appropriate choice of conformal compactification is crucial to guaranteeing a maximal regularity result. Here, we introduce the constant scalar curvature compactification.

Proposition 2.3. *Let $U \subset \overline{M}$ be an open set with boundary portion D and let g_+ be $C^{m,\sigma}$ conformally compact on $U \cap M$, where $m \geq 2$ and $0 < \sigma < 1$. Let $h \in C^{m,\sigma}(D)$ be an element of the conformal infinity for g_+. Then near any point $p \in D$, there is a defining function ρ such that $g = \rho^2 g_+ \in C^{m,\sigma}$ near p with the properties that g has boundary metric h and scalar curvature constantly equal to $-n(n-1)$. Moreover, if $\hat{\rho}$ is a $C^{m+1,\sigma}$ defining function such that $\hat{\rho}^2 g_+$ restricts to h then $\rho/\hat{\rho} \in C^{m,\sigma}$ near p.*

Proof. We start by constructing a set similar to that used in the proof of Proposition 2.1. In particular, we consider an open set $V \subset U \cap M$ with the following properties. First, $\overline{V} \subset U$. Second, V is the interior of a C^∞ manifold with boundary, diffeomorphic to an open n-dimensional ball. Third, $\partial V \cap \partial M$ is a subset of D diffeomorphic to a closed $(n-1)$-dimensional ball in ∂M, and contains p in its interior. With this in place, let $\hat{g} = \hat{\rho}^2 g_+$ be a compactification with boundary metric h, where $\hat{\rho} \in C^{m+1,\sigma}(\overline{V})$ and $\hat{g} \in C^{m,\sigma}$. Our goal is to find a positive function v on $\overline{V}$ such that $g = v^{\frac{4}{n-2}} \hat{g}$ has constant scalar curvature. Looking at how scalar curvature changes under this conformal transformation, we have

$$S = \hat{S}\, v^{-\left(\frac{4}{n-2}\right)} + \frac{4(1-n)}{n-2} v^{-\left(\frac{n+2}{n-2}\right)} \hat{\Delta} v. \tag{2}$$

Setting $S = -n(n-1)$ (any negative number will work) we may reduce our problem to the following:

$$\begin{cases} \hat{\Delta} v - \frac{(n-2)\hat{S}}{4(n-1)} v - \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0 \\ v|_{\partial V} = 1. \end{cases}$$

This is a special case of the Yamabe problem on a manifold with boundary and always has a $C^{2,\sigma}$ solution [Ma95]. Moreover, since the given differential operator is elliptic, for $\hat{g} \in C^{m,\sigma}$, a bootstrap procedure shows that any such solution v is in $C^{m,\sigma}(\overline{V})$.

With this, we can construct a new defining function $\rho = v^{\frac{2}{n-2}} \hat{\rho}$, with which to compactify. Note that since $v = 1$ on $D \cap \partial V$, the boundary metric h is not changed near p. □

We remark that Proposition 2.3 has a global analogue. Namely, if g_+ is $C^{m,\sigma}$ conformally compact on M then there is a constant scalar curvature compactification $\rho^2 g_+$ in $C^{m,\sigma}(\overline{M})$ and for any $C^{m+1,\sigma}$ defining function $\hat{\rho}$ for ∂M, we have $\rho/\hat{\rho} \in C^{m,\sigma}(\overline{M})$. This follows by using M in place of V in the proof above.

If g_+ is Einstein and we work in harmonic coordinates for the constant scalar curvature compactification g, then interior regularity for g is significantly better than that given in Proposition 2.3. In fact, we have the following result. Note that ∂M is not involved, and ρ need not be a defining function.

Proposition 2.4. *Let V be an open subset of M and let g_+ be an Einstein metric in $C^2(V)$. Let $g = \rho^2 g_+$ have constant scalar curvature, where $\rho \in C^2(V)$ is a positive function and let $W \subset V$ be an open set on which harmonic coordinates for g are defined. Then in these coordinates, g and ρ are both in $C^\infty(W)$.*

Proof. Writing out the equation $Ric_+ = -(n-1)g_+$ in terms of g, we find

$$Ric_{\alpha\beta} = \mathcal{P}(g^{-1}, \partial g, \rho^{-1}, \partial^2 \rho).$$

Since we are in harmonic coordinates, we can use (1) and rearrange to get

$$\mathcal{L} g_{\alpha\beta} = \mathcal{P}(g^{-1}, \partial g, \rho^{-1}, \partial^2 \rho). \tag{3}$$

Also, similar to (2), we have

$$S_+ = S\rho^2 + 2(n-1)\rho\Delta\rho - n(n-1)|d\rho|_g^2.$$

By isolating the second derivatives of ρ, this equation reduces to

$$\mathcal{L}\rho = \mathcal{P}(g^{-1}, \partial g, \rho^{-1}, \partial \rho) \tag{4}$$

since the scalar curvatures of g_+ and g are both constant.

Now we run a double bootstrap to get the result. To start, note that in harmonic coordinates for g, the function ρ and the components of g are all in $C^{1,\sigma}(W)$. Proceeding by induction, let $m \geq 1$ and suppose that ρ and the components of g are all in $C^{m,\sigma}(W)$. Then the right hand side of (4) is in $C^{m-1,\sigma}(W)$, so by elliptic regularity, we may conclude that $\rho \in C^{m+1,\sigma}(W)$. Using this, we then observe that the right hand side of (3) is in $C^{m-1,\sigma}(W)$, and so we have $g_{\alpha\beta} \in C^{m+1,\sigma}(W)$. Therefore, by induction it follows that $g_{\alpha\beta}$ and ρ are in $C^{m,\sigma}(W)$ for all m. □

3 Almost Geodesic Coordinates for an Almost Geodesic Compactification

Geodesic coordinates provide a means of expressing the Taylor expansion at the boundary of a geodesic compactification of a conformally compact Einstein metric in terms of the boundary metric. Unfortunately, using exact geodesic coordinates for an exact geodesic compactification results in a loss of regularity. Here, we construct approximate versions of the geodesic compactification and geodesic coordinates which avoid the loss of regularity. The approximate geodesic coordinates that we use are the approximate Gaussian coordinates introduced in [AC96] in the case when the background metric is taken to be our "almost geodesic compactification". Also see [CDLS05] for similar constructions.

3.1 Geodesic Compactification and Geodesic Coordinates

We start with a review of the geodesic compactification and its associated geodesic coordinates. This compactification and a number of its properties were introduced in [GL91].

3.1.1 Geodesic Compactification

Given an asymptotically hyperbolic metric g_+ on a coordinate domain $U \subset \overline{M}$ with boundary portion D, it is possible to find a defining function $\tilde{\rho}$, with associated compactified metric $\tilde{g}$, so that $|d\tilde{\rho}|_{\tilde{g}} = 1$ not just on D, but on a neighborhood of the boundary. Moreover, given a boundary metric h in the conformal infinity of g_+, $\tilde{\rho}$ can be chosen so that the new compactification has h as its boundary metric. To do this, let g be a compactification for g_+, with defining function ρ, such that its boundary metric is h. To determine $\tilde{g}$, write $\tilde{\rho} = e^u\rho$. Then we want to find u such that $u|_{\partial M} = 0$ and $|d(e^u\rho)|^2_{e^{2u}g} = 1$ in U. We may take boundary adapted coordinates $\{x^\alpha\}$ for U, and since ρ is a C^∞ defining function, we may take it to be x^0. In this setting, the constant length condition gives:

$$2g^{0\alpha}u_\alpha + \rho|du|^2_g = \frac{1 - |d\rho|^2_g}{\rho}. \tag{5}$$

We are guaranteed a solution to this equation by general theory of first order partial differential equations. As for regularity, if g_+ is C^m asymptotically hyperbolic, we know that the right hand side of this equation is $C^m(U)$ and so the best we can say for an exact solution u to this differential equation is that $u \in C^m(U)$. Hence $\tilde{\rho} \in C^m(U)$ as well. See [Lee95] for a detailed discussion of Hölder regularity results for this equation. We call this compactification a *geodesic compactification*, and the associated defining function is called a *geodesic defining function*. The terminology here comes from the fact that the integral curves of $\mathrm{grad}_{\tilde{g}}(\tilde{\rho})$ are geodesics.

3.1.2 Geodesic Coordinates

The geodesic defining function $\tilde{\rho}$ can be used to identify points near D with elements of $D \times [0, \varepsilon)$ and to split $\tilde{g}$ as $\tilde{g}_{\tilde{\rho}} + d\tilde{\rho}^2$, where $\tilde{g}_{\tilde{\rho}}$ is a one parameter family of metrics on D. To achieve this, choose any coordinate system $\{\tilde{x}^i; i = 1, \ldots, n-1\}$ on D. Extend these coordinates inward by keeping them constant on the integral curves of $\mathrm{grad}_{\tilde{g}}(\tilde{\rho})$. These functions taken together with $\tilde{x}^0 = \tilde{\rho}$ form a boundary adapted coordinate system. Such coordinates are called *geodesic coordinates* for $\tilde{g}$.

Regularity of $\tilde{g}$ is not preserved in its geodesic coordinates. On the other hand, $\tilde{g}_{00} = 1$ and $\tilde{g}_{i0} = 0$ by construction. If $\tilde{g}$ is a geodesic compactification for an Einstein metric g_+, then we can say more. In particular, when $n \geq 4$ is even and $\tilde{g}$ is sufficiently smooth with respect to its geodesic coordinates, then for $0 \leq p \leq n-2$,

$$(\partial_0^p \tilde{g}_{ij})|_D = \mathcal{P}(h^{-1}, \partial_t^p h).$$

Moreover, if p is odd, then in fact $(\partial_0^p \tilde{g}_{ij})|_D = 0$. It follows immediately that if we allow for derivatives with respect to $\tilde{x}^i$, then

$$(\partial^p \tilde{g}_{ij})|_D = \mathcal{P}(h^{-1}, \partial_t^p h). \tag{6}$$

These expressions, along with similar expressions when n is odd, are derived in [Gra00].

3.2 Almost Geodesic Compactification and Almost Geodesic Coordinates

We now construct approximate versions of the geodesic compactification and geodesic coordinates, which preserve the useful expressions given above while avoiding the loss of regularity inherent in exact geodesic coordinates.

3.2.1 Error Terms

Because we will be working with approximations, we introduce some notation for the error terms that arise. In a coordinate domain U, by $\mathcal{E}_m$ we will denote a function which is in $C^m(U)$, and which is $o(\rho^m)$ for some defining function ρ. Note that a C^m function f is $\mathcal{E}_m$ if and only if all derivatives up to order m are zero on the boundary. Different instances of this notation will not mean the same function.

3.2.2 Almost Geodesic Defining Function and Almost Geodesic Compactification

Our first step in producing an appropriate approximation to the geodesic compactification is to construct the right defining function. For a C^m asymptotically hyperbolic metric g_+, an exact solution to (5) is C^m. If we allow an approximate solution, we can do a bit better. We say a function $\tilde{\rho}$ is an *almost geodesic defining function of order m* for D if $\tilde{\rho}$ is a defining function for D in $C^{m+1}(U) \cap C^\infty(U \cap M)$ and $1 - |d\tilde{\rho}|^2_{\tilde{\rho}^2 g_+} = \tilde{\rho}\,\mathcal{E}_m$. Construction of a global almost geodesic defining function of order 2 can be found in [CDLS05]. For higher orders and in the local setting, the following lemma guarantees that such a function exists by using an extension result in [AC96] in an inductive argument.

Lemma 3.1. *Let g_+ be a C^m asymptotically hyperbolic metric on a coordinate domain U with boundary portion D, and let $h \in C^m(D)$ be an element of the conformal infinity for g_+. Then there exists an almost geodesic defining function $\tilde{\rho}$ of order m such that the boundary metric of $\tilde{\rho}^2 g_+$ is h.*

Proof. Let $g = \rho^2 g_+$ be a C^m compactification that restricts to h on the boundary, where ρ is a C^∞ defining function. Taking boundary adapted coordinates $\{x^\alpha\}$ with $x^0 = \rho$, our goal is to find a function u that solves (5) approximately. Let f be the right hand side of (5), and note that since g_+ is C^m asymptotically hyperbolic on U, $f \in C^m(U)$. Formally differentiating (5) $l-1$ times with respect to ρ, $1 \le l \le m+1$ and setting ρ equal to zero we obtain

$$(\partial_0^l u)|_D = \mathcal{P}_l$$

where $\mathcal{P}_l = \mathcal{P}(g^{-1}, (g^{00})^{-1}, \partial_0^{l-1} g, \partial_0^{l-1} f, \partial_t \partial_0^{l-1} u)$.

Hence, starting with $v^0 = 0$, we can recursively define functions $v^l = \mathcal{P}_l \in C^{m-l+1}(D)$. By Corollary 3.3.2 in [AC96], there is a function $v \in C^{m+1}(U) \cap C^\infty(U \cap M)$ such that $(\partial_0^l v)|_D = v^l$.

Now, by construction v solves (5) modulo $\mathcal{E}_m$, so defining $\tilde{\rho} = e^v \rho$, the result follows. □

Given a C^m asymptotically hyperbolic metric g_+, we say that a metric $\tilde{g}$ is an *almost geodesic compactification of order m associated to g_+* if $\tilde{g} = \tilde{\rho}^2 g_+$, where $\tilde{\rho}$ is an almost geodesic defining function of order m.

3.2.3 Almost Geodesic Coordinates

Given a C^m asymptotically hyperbolic metric g_+ and an almost geodesic compactification $\tilde{g} = \tilde{\rho}^2 g_+$ of order m associated to g_+, we say a boundary adapted coordinate system $\{\tilde{x}^\alpha\}$ is an *almost geodesic coordinate system of order m* for $\tilde{g}$ if $\tilde{x}^i \in C^{m+1}$ with respect to the given smooth structure, $\tilde{x}^0 = \tilde{\rho}$, and in these coordinates, $\tilde{g}_{00} = 1 + \mathcal{E}_m$ and $\tilde{g}_{i0} = \mathcal{E}_m$. Note that the components of $\tilde{g}$ are all C^m in this coordinate system. As for existence, we have the following:

Lemma 3.2. *Let g_+ be a C^m asymptotically hyperbolic metric on a coordinate domain U containing boundary portion D and let $\tilde{g} = \tilde{\rho}^2 g_+$ be an almost geodesic compactification of order m associated to g_+. Let $\{x^i, i = 1, \ldots, n-1\}$ be coordinates for D, which are C^{m+1} compatible with the given smooth structure. Then there is an almost geodesic coordinate system of order m for $\tilde{g}$ such that the restriction of the tangential coordinates are the given boundary coordinates.*

Proof. Proposition B.1.1 in [AC96] provides us with coordinates $\{\tilde{x}^\alpha\}$ which are C^{m+1} with respect to the given smooth structure and such that $\tilde{g}_{00} = 1 + \mathcal{E}_m$ and $\tilde{g}_{i0} = \mathcal{E}_m$. Moreover, they can be chosen so that $\tilde{x}^i = x^i$ on the boundary.

We now work in this coordinate system to show that $\tilde{x}^0$ and $\tilde{\rho}$ agree modulo $\mathcal{E}_{m+1}$. We already know that $\tilde{x}^0$ agrees with $\tilde{\rho}$ on the boundary, since they are both defining functions. Moreover, since $\tilde{\rho}$ is an almost geodesic defining function for $\tilde{g}$, we have

$$1 - \tilde{g}^{\alpha\beta}\tilde{\rho}_{,\alpha}\,\tilde{\rho}_{,\beta} = \tilde{\rho}\,\mathcal{E}_m. \tag{7}$$

Evaluating this at the boundary, we find that $\tilde{\rho}_{,0} = 1$. To determine higher derivatives at the boundary, we differentiate (7) l times with respect to $\tilde{x}^0$, $1 \leq l \leq m$ to get

$$(\partial_0^l \tilde{g}^{\alpha\beta})\tilde{\rho}_{,\alpha}\,\tilde{\rho}_{,\beta} + 2\tilde{g}^{\alpha\beta}(\partial_0^l \tilde{\rho}_{,\alpha})\tilde{\rho}_{,\beta} + \mathcal{P}(\tilde{g}^{-1}, \partial_0^{l-1}\tilde{g}, \partial^l \tilde{\rho}) = (\partial_0^l \tilde{\rho})\mathcal{E}_m + \cdots + (\partial_0 \tilde{\rho})\mathcal{E}_{m-l+1} + \tilde{\rho}\,\mathcal{E}_{m-l},$$

where in this case, every term in the polynomial $\mathcal{P}$ on the left hand side includes at least one derivative of order ≥ 2 of $\tilde{\rho}$. Working inductively, and using the form of $\tilde{g}$ in the given coordinates, we find that at the boundary, this equation reduces to

$$\partial_0^{l+1}\tilde{\rho} = 0.$$

Hence, $\tilde{\rho} = \tilde{x}^0 + \mathcal{E}_{m+1}$, and therefore the form of the metric only changes modulo $\mathcal{E}_m$ when we look at the change of coordinates from $\tilde{x}^0$ to $\tilde{\rho}$. □

If g_+ is an asymptotically hyperbolic Einstein metric and $\tilde{g}$ is an associated almost geodesic compactification, then in an almost geodesic coordinate system for $\tilde{g}$ we get an expansion for the tangential components of $\tilde{g}$ at the boundary similar to that for geodesic coordinates.

Proposition 3.3. *Let $2 \leq m \leq n-2$ and let g_+ be a C^m asymptotically hyperbolic Einstein metric on a coordinate domain $U \subset \overline{M}$ with boundary portion $D \subset \partial M$. Let $\tilde{g}$ be an almost geodesic compactification associated to g_+ with boundary metric h, and let $\{\tilde{\rho}, \tilde{x}^i\}$ be almost geodesic coordinates of order m for $\tilde{g}$ on U. Then for $0 \leq p \leq m$ we have*

$$(\partial^p \tilde{g}_{ij})|_D = \mathcal{P}(h^{-1}, \partial_t^p h), \tag{8}$$

where the polynomials on the right hand side are the same polynomials which appear in (6). *In particular, $(\partial_0^p \tilde{g}_{ij})|_D = 0$ when p is odd.*

Proof. The derivation of this expression follows the methods used in [Gra00] to generate the similar expansion for a geodesic compactification in geodesic coordinates. We note that if we can verify (8) for $(\partial_0^p \tilde{g}_{ij})|_D$, then the more general result follows immediately by taking derivatives with respect to $\tilde{x}^i$ on the boundary.

Writing out the condition $Ric_+ = -(n-1)g_+$ under the conformal transformation $\tilde{g} = \tilde{\rho}^2 g_+$ and focusing on the tangential components, we find that in our almost geodesic coordinates,

$$\begin{aligned} &\tilde{\rho}\,\tilde{g}_{ij,00} + (2-n)\tilde{g}_{ij,0} - \tilde{g}^{kl}\tilde{g}_{kl,0}\,\tilde{g}_{ij} \\ &\qquad - \tilde{\rho}\,\tilde{g}^{kl}\tilde{g}_{ik,0}\,\tilde{g}_{jl,0} + \frac{\tilde{\rho}}{2}\,\tilde{g}^{kl}\tilde{g}_{kl,0}\,\tilde{g}_{ij,0} - 2\tilde{\rho}\,\hat{Ric}_{ij} = \tilde{\rho}\,\mathcal{E}_{m-2}, \end{aligned} \tag{9}$$

where $\hat{Ric}$ is the Ricci tensor for the induced metric on level sets of $\tilde{\rho}$. This equation is the same as (2.5) in [Gra00], except for the $\tilde{\rho}\,\mathcal{E}_{m-2}$ term arising from the error terms on $\tilde{g}$ and its first and second derivatives. For $1 \leq p \leq m-1$, we differentiate (9) $p-1$ times with respect to $\tilde{\rho}$ to get

$$\begin{aligned} &\tilde{\rho}\,\partial_0^{p+1}\tilde{g}_{ij} + (p-n+1)\partial_0^p \tilde{g}_{ij} - \tilde{g}^{kl}(\partial_0^p \tilde{g}_{kl})\tilde{g}_{ij} \\ &\qquad = \tilde{\rho}\,\mathcal{P}(\tilde{g}^{-1}, \partial_0^p \tilde{g}, \partial_0^{p-2}\partial_t^2 \tilde{g}) + \mathcal{P}(\tilde{g}^{-1}, \partial_0^{p-1}\partial_t^2 \tilde{g}) + \tilde{\rho}\,\mathcal{E}_{m-p-1}. \end{aligned} \tag{10}$$

Setting $\tilde{\rho}$ equal to zero, tracing with respect to h, and then plugging back in, we recursively solve for $(\partial_0^p \tilde{g}_{ij})|_D$ to get (8) for $1 \leq p \leq m-1$.

We cannot differentiate (9) any more than this because it would generate derivatives of $\tilde{g}$ of order $m+1$. Instead, to derive (8) in the case of m derivatives, we set $p = m-1$ in (10), divide by $\tilde{\rho}$, and

evaluate the result as a limit of a difference quotient. To do this, we need to know something about the derivative of order $m-1$. Evaluating (10) at $\tilde{\rho}=0$, we have

$$(m-n)\partial_0^{m-1}\tilde{g}_{ij} - \tilde{g}^{kl}(\partial_0^{m-1}\tilde{g}_{kl})\tilde{g}_{ij} = (\mathcal{P}_{m-1})|_D, \tag{11}$$

where $\mathcal{P}_{m-1} = \mathcal{P}(\tilde{g}^{-1}, \partial_0^{m-2}\partial_t^2\tilde{g})$ is the second polynomial term on the right hand side of (10). Now, dividing (10) by $\tilde{\rho}$ and rearranging we have

$$\begin{aligned} \partial_0^m \tilde{g}_{ij} + \left(\frac{(m-n)\partial_0^{m-1}\tilde{g}_{ij} - \tilde{g}^{kl}(\partial_0^{m-1}\tilde{g}_{kl})\tilde{g}_{ij} - \mathcal{P}_{m-1}}{\tilde{\rho}} \right) & \\ = \quad & \mathcal{P}(\tilde{g}^{-1}, \partial_0^p\tilde{g}, \partial_0^{p-2}\partial_t^2\tilde{g}) + \mathcal{E}_0. \end{aligned}$$

We then take the limit as $\tilde{\rho}\to 0$. The difference quotient converges since, at the boundary, the first two terms converge to the third term by (11). Moreover, the limit is exactly what we would have gotten if we were to differentiate each term. Hence, we can solve for $\partial_0^m\tilde{g}_{ij}$ as above and when we do so, the result is the same as what we would get by formally differentiating.

As a final note, observe that even if $\tilde{g}$ is smoother than C^{n-2}, we cannot make use of (10) for $p\geq n-1$ because when $p=n-1$, the second term on the left hand side is zero. Without this term, we can no longer recursively solve for the derivative of $\tilde{g}$. □

3.2.4 Curvature of an Almost Geodesic Compactification

The fact that the derivatives at the boundary of an almost geodesic compactification $\tilde{g}$ can be expressed in terms of its boundary metric allows us to simplify coordinate expressions at the boundary for the Ricci tensor and scalar curvature tensor of $\tilde{g}$, along with their covariant derivatives.

Lemma 3.4. *Let $2\leq m\leq n-2$ and let $\tilde{g}$ be an almost geodesic compactification of order m associated to a C^m asymptotically hyperbolic Einstein metric. In almost geodesic coordinates of order m and for $0\leq p\leq m-2$, we have*

$$(\tilde{\nabla}^p\tilde{Ric})|_{\partial M} = \mathcal{P}(h^{-1}, \partial_t^{p+2}h)$$

and

$$(\tilde{\nabla}^p\tilde{S})|_{\partial M} = \mathcal{P}(h^{-1}, \partial_t^{p+2}h).$$

Proof. Observe that components of $\tilde{\nabla}^p\tilde{Ric}$ involve at most $p+2$ derivatives of $\tilde{g}$. Since we are in almost geodesic coordinates for $\tilde{g}$, we can use (8) along with the facts that $\tilde{g}_{00}=1+\mathcal{E}_m$ and $\tilde{g}_{0i}=\mathcal{E}_m$ to conclude that $(\tilde{\nabla}^p\tilde{Ric})|_{\partial M}$ can be expressed completely in terms of h and its first $p+2$ derivatives. This is valid as long we are taking few enough derivatives that we can make use of (8). For this, we require $p+2\leq m$. The same analysis gives us the result for $\tilde{S}$ and its covariant derivatives. □

We will ultimately be working in coordinates that are not necessarily almost geodesic coordinates. The next lemma shows that we can work with different boundary adapted coordinates without losing the important characterization of curvature tensors and their derivatives from the above lemma. For this lemma, and in the future, we make use of the inward pointing normal vector $\mathcal{N}$ relative to a given metric g. Note that in boundary adapted coordinates $\{x^\alpha\}$, $\mathcal{N}$ can be written in terms of the coordinate vectors $\partial/\partial x^\alpha$:

$$\mathcal{N} = \frac{\operatorname{grad}_g(x^0)}{|dx^0|_g} = \frac{g^{\alpha 0}}{\sqrt{g^{00}}}\frac{\partial}{\partial x^\alpha}. \tag{12}$$

Lemma 3.5. *Let $2\leq m\leq n-2$ and let $\tilde{g}$ be an almost geodesic compactification of order m associated to a C^m asymptotically hyperbolic Einstein metric. In any arbitrary boundary adapted coordinate system that is C^{m+1} compatible with the given smooth structure and for $0\leq p\leq m-2$, we have*

$$(\tilde{\nabla}^p\tilde{Ric})|_{\partial M} = \mathcal{P}(\tilde{g}^{-1}, (\tilde{g}^{00})^{-\frac{1}{2}}, \partial_t^{p+2}h)$$

and

$$(\tilde{\nabla}^p\tilde{S})|_{\partial M} = \mathcal{P}(\tilde{g}^{-1}, (\tilde{g}^{00})^{-\frac{1}{2}}, \partial_t^{p+2}h).$$

Proof. Let $\{x^\alpha\}$ be a boundary adapted coordinate system that is C^{m+1} compatible with the given smooth structure. By Lemma 3.2, there is an almost geodesic coordinate system $\{\tilde{x}^\alpha\}$ of order m for $\tilde{g}$ such that $\tilde{x}^i|_D = x^i|_D$. In what follows, the metric and other tensors are being expressed in terms of $\{x^\alpha\}$, so in order to use the expressions in Lemma 3.4 we need to know how to express the coordinate vectors $\partial/\partial x^\alpha$ in terms of $\partial/\partial \tilde{x}^\alpha$ at the boundary. Note that by construction, we already have $\partial/\partial x^i = \partial/\partial \tilde{x}^i$ on ∂M. Moreover, $\partial/\partial \tilde{x}^0$ is the inward pointing normal vector $\tilde{\mathcal{N}}$. By rearranging (12), we can solve for $\partial/\partial x^0$ in terms of $\partial/\partial \tilde{x}^\alpha$ at the boundary:

$$\frac{\partial}{\partial x^0} = \frac{\tilde{\mathcal{N}}}{\sqrt{\tilde{g}^{00}}} - \frac{\tilde{g}^{i0}}{\tilde{g}^{00}}\frac{\partial}{\partial x^i} = \frac{1}{\sqrt{\tilde{g}^{00}}}\frac{\partial}{\partial \tilde{x}^0} - \frac{\tilde{g}^{i0}}{\tilde{g}^{00}}\frac{\partial}{\partial \tilde{x}^i}.$$

With this, we compute

$$\begin{aligned}(\tilde{\nabla}^p \tilde{Ric})&\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^{\mu_1}}, \frac{\partial}{\partial x^{\mu_2}}, \ldots, \frac{\partial}{\partial x^{\mu_p}}\right) \\ &= \quad F^{\mu\delta\eta_1\eta_2\ldots\eta_p}_{\alpha\beta\mu_1\mu_2\ldots\mu_p}\,\tilde{\nabla}^p \tilde{Ric}\left(\frac{\partial}{\partial \tilde{x}^\mu}, \frac{\partial}{\partial \tilde{x}^\delta}, \frac{\partial}{\partial \tilde{x}^{\eta_1}}, \frac{\partial}{\partial \tilde{x}^{\eta_2}}, \ldots, \frac{\partial}{\partial \tilde{x}^{\eta_p}}\right)\end{aligned}$$

where $F^{\eta_1\eta_2\cdots\eta_p}_{\mu_1\mu_2\cdots\mu_p} = \mathcal{P}(\tilde{g}^{-1}, (\tilde{g}^{00})^{-\frac{1}{2}})$, by the above facts about coordinate vectors. The covariant derivative of Ricci on the right hand side can then be replaced by the polynomial in Lemma 3.4 to get the result. ☐

Lemma 3.5 gives us formulas for the Ricci tensor and scalar curvature of an almost geodesic compactification at the boundary in arbitrary boundary adapted coordinates, but they are not explicit. Explicit formulas for the Ricci tensor and scalar curvature in dimension 4 are presented in [And03], and the formula for scalar curvature in general dimension is provided in [Lee95].

We finish this section with a consideration of the second fundamental form $\tilde{A}$ of the boundary. Our convention here, and in the future, is that the normal vector used to define the second fundamental form points inward.

Lemma 3.6. *Let $\tilde{g}$ be an almost geodesic compactification for a C^2 asymptotically hyperbolic Einstein metric. Then $\tilde{A} = 0$ on D.*

Proof. Since the second fundamental form is a tensor, we can prove this result in coordinates of our choice, so we use almost geodesic coordinates $\{\tilde{x}^\alpha\}$. Since these are boundary adapted coordinates, we have

$$\tilde{A}_{ij} = (\tilde{g}^{00})^{-\frac{1}{2}}\,\tilde{\Gamma}^0_{ij},$$

Writing out the Christoffel symbol and using the form of $\tilde{g}$ in its almost geodesic coordinates, along with the fact that $(\partial_0 \tilde{g}_{ij})_D = 0$ by Proposition 3.3, we have the result. ☐

4 Deriving the Boundary Value Problem

In this section, we derive a boundary value problem for components of a compactification of an asymptotically hyperbolic Einstein metric. The compactification we use will not be an almost geodesic compactification, but rather one which has constant scalar curvature. Nonetheless, an almost geodesic compactification will come into the picture as a tool for deriving boundary equations for the system.

Proposition 4.1. *Let $\overline{M}$ be an n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. Let $U \subset \overline{M}$ be a coordinate domain with boundary portion $D = U \cap \partial M$. If $n > 4$, let g_+ be a C^{n-2} asymptotically hyperbolic Einstein metric on $U \cap M$, and if $n = 4$, let g_+ be a C^3 conformally compact Einstein metric on $U \cap M$. Let $h \in C^{n-2}(D)$ ($C^3(D)$ if $n = 4$) be an element of the conformal infinity of g_+ and suppose that g_+ has a constant scalar curvature compactification $g \in C^{n-2}(U)$ ($C^3(U)$ if $n = 4$) with scalar curvature $S = -n(n-1)$ and boundary metric h. Finally, suppose that g has boundary adapted harmonic coordinates $\{x^\alpha\}$ in $C^{n-1}(U)$ ($C^4(U)$ if $n = 4$). Then, in these harmonic coordinates, the components of g solve a system and accompanying boundary equations of the following form:*

- *On $U \cap M$:*

$$\mathcal{L}_{\frac{n}{2}} g_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial^{n-1} g) = 0. \tag{13}$$

- *On D:*
  - *Equations of order 0: For $1 \le i, j \le n-1$,*

$$g_{ij} = h_{ij}. \tag{14}$$

  - *Equations of order 1: For $0 \le \alpha \le n-1$,*

$$g^{\eta\beta} \partial_\eta g_{\alpha\beta} - \frac{1}{2} g^{\eta\beta} \partial_\alpha g_{\eta\beta} = 0. \tag{15}$$

  - *Equations of order 2: For $1 \le i, j \le n-1$,*

$$\mathcal{L} g_{ij} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h) = 0. \tag{16}$$

  - *Equations of order 3: For $0 \le \alpha \le n-1$,*

$$g^{\eta\beta} \partial_\eta \mathcal{L} g_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial^2 g) = 0. \tag{17}$$

  - *Equations of order $2l$, $2 \le l \le (n/2) - 1$: For $0 \le \alpha, \beta \le n-1$,*

$$\mathcal{L}_l g_{\alpha\beta} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{2l-1} g, \partial_t^{2l} h) = 0. \tag{18}$$

Additionally, the following formula for the derivatives of the second fundamental form holds on D. For $1 \le k \le n-1$,

$$\partial_k A_{ij} = \frac{(g^{00})^{\frac{1}{2}} g_{ij}}{2(2-n)} \mathcal{L} g_{k0} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h). \tag{19}$$

Before focusing on the derivation of (13)–(19), there are a number of comments to be made with regard to Proposition 4.1. First, by Proposition 2.4, we know that g is C^∞ in its harmonic coordinates on the interior, so (13) makes sense classically. Second, the reason for the stronger a priori regularity in dimension four is to accommodate the boundary equations of order 3. Third, we note that the regularity of g and h are preserved when moving to the given harmonic coordinates, since the coordinates are one degree smoother than g and h. Fourth, observe that for the equations of order greater than 3, there is no difference in behavior between the tangential and non-tangential components of g. The reason for this will become clear in the course of the proof. Finally, (19) is part of the boundary value problem in that it provides important relations among the second derivatives of g. As a consequence, it will play a crucial role in the proof of boundary regularity for the system. In section 7, we will see an alternative way to incorporate (19) into the rest of the boundary data.

To prove Proposition 4.1, we start by introducing the so called Ambient Obstruction Tensor $\mathcal{O}$ and we recall that it vanishes for any metric which is conformal to an Einstein metric. The equation $\mathcal{O}_{\alpha\beta} = 0$ then reduces to (13). Moving on to the boundary equations, we will find that the equations of order 1 and 3 follow as restrictions to the boundary of equations that hold on the interior. The remaining equations, including (19), will not come as naturally. They essentially arise from the fact that through an almost geodesic compactification, powers of the Laplacian of the Ricci tensor can be expressed in terms of boundary data, modulo lower order terms.

4.1 The System via the Ambient Obstruction Tensor

In [FG85] (Proposition 3.5), Fefferman and Graham introduced a generalization of the Bach tensor in each even dimension called the *Ambient Obstruction Tensor $\mathcal{O}$*. In indices, it has the following form:

$$\mathcal{O}_{\alpha\beta} = \frac{1}{(-2)^{\frac{n}{2}-2}(\frac{n}{2}-2)!} \left(\Delta^{\frac{n}{2}-1} P_{\alpha\beta} - \frac{1}{2(n-1)} \Delta^{\frac{n}{2}-2} S_{,\alpha\beta} \right) + \mathcal{P}(g^{-1}, \partial^{n-1} g) \tag{20}$$

where

$$P_{\alpha\beta} = \frac{1}{n-2} \left(Ric_{\alpha\beta} - \frac{1}{2(n-1)} S\, g_{\alpha\beta} \right).$$

When $n = 4$ this tensor is the Bach tensor, and in general, it is symmetric, trace free, conformally invariant with weight $(n-2)/2$, and equal to zero when the metric is Einstein. See [GH05] for details surrounding these facts.

Proof of the System Equations. Since we are in harmonic coordinates for a C^2 constant scalar curvature compactification for an Einstein metric, $g \in C^\infty(U \cap M)$ by Proposition 2.4, so the ambient obstruction tensor is well defined. Moreover, (1) can be used for Ricci and S is constant, so (20) reduces to

$$\mathcal{O}_{\alpha\beta} = \frac{1}{(-2)^{\frac{n}{2}-1}(\frac{n}{2}-2)!(n-2)} \left(\mathcal{L}_{\frac{n}{2}} g_{\alpha\beta}\right) + \mathcal{P}(g^{-1}, \partial^{n-1} g).$$

From the properties of the ambient obstruction tensor above, this is equal to zero since g is conformally Einstein. After multiplying by the leading constant, we have (13). □

4.2 Boundary Equations of Order 0, 1, and 3

As mentioned, these equations are almost immediate.

Proof of the Boundary Equations of Order 0, 1, and 3. For the equations of order 0, we have (14) since g restricts to h on D and we are working in boundary adapted coordinates.

The equations of order 1 follow from the statement that the given coordinates are harmonic with respect to g. The equation $\Delta x^\eta = 0$ is equivalent to $g^{\alpha\beta}\Gamma^\eta_{\alpha\beta} = 0$. Writing out the Christoffel symbol in terms of derivatives of the metric, lowering the free index, and relabeling, we obtain (15).

Finally, the equations of order 3 follow from the contracted Bianchi identity, the fact that S is constant, and (1). We have

$$0 = \frac{1}{2} S_{,\alpha} = \nabla^\beta Ric_{\alpha\beta} = g^{\eta\beta}\left(-\frac{1}{2}(\mathcal{L} g_{\alpha\beta})_{,\eta} + \mathcal{P}(g^{-1}, \partial^2 g)\right),$$

and (17) follows. □

4.3 Boundary Equations of Order $2l$ and Equation (19)

For these equations, we relate tensors associated with g to tensors associated to an almost geodesic compactification of order $n-2$ which is conformally related to g. The tensors associated with g give rise to the operator $\mathcal{L}_l$ acting on the components of the metric and, through Lemma 3.5, the related tensors for the almost geodesic metric can be expressed in terms of boundary data. The bulk of the work is in analyzing the conformal factor relating g and the almost geodesic compactification.

4.3.1 Boundary Equations of Order 2 and Equation (19)

To generate the boundary equations of order 2, we will look at the way that the Ricci tensor changes under a conformal transformation to an almost geodesic metric. This will give us a formula for $\mathcal{L}$ acting on components of g, but will also introduce derivatives of the conformal factor. We will deal with these by analyzing the second fundamental form and the scalar curvature under the conformal change.

By Lemma 3.1, we can choose an almost geodesic compactification $\tilde{g}$ of order $n-2$ for g_+ such that $\iota^*\tilde{g} = \iota^* g = h$. Then we have $g = \phi^2 \tilde{g}$ with $\phi \in C^{n-2}(U)$ and $\phi|_D = 1$. In this and subsequent sections, we will be passing back and forth between these two metrics, so we observe that when taking derivatives we have

$$\partial^p g = \mathcal{P}(\partial^p \phi, \partial^p \tilde{g})$$

and

$$\partial^p \tilde{g} = \mathcal{P}(\phi^{-1}, \partial^p \phi, \partial^p g).$$

We start with some lemmas that tell us how to deal with derivatives of ϕ.

Lemma 4.2. *The first derivatives of ϕ at the boundary are as follows:*

$$(\phi_{,i})|_D = 0$$

and

$$(\phi_{,0})|_D = \mathcal{P}(g^{-1}, (g^{00})^{-1}, \partial g). \tag{21}$$

Proof. The tangential derivatives are immediate since ϕ is constant on the boundary. For the transverse derivative, we use the second fundamental form. Looking at the way that the second fundamental form changes under our conformal change, we have

$$\tilde{A}_{ij} = \phi^{-1} A_{ij} + \phi^{-2} \phi_{,\alpha} \mathcal{N}^{\alpha} g_{ij}.$$

We know that $\phi|_D = 1$ and $(\phi_i)|_D = 0$. Moreover, $\tilde{A}$ is zero by Lemma 3.6 and $\mathcal{N}^{\alpha} = g^{\alpha 0}(g^{00})^{-\frac{1}{2}}$ by (12). Hence this formula simplifies substantially to

$$A_{ij} = -\phi_{,0}\,(g^{00})^{\frac{1}{2}} g_{ij}. \tag{22}$$

We are in boundary adapted coordinates so $A_{ij} = (g^{00})^{-\frac{1}{2}}\,\Gamma^0_{ij}$. Plugging this in, along with the fact that $\Gamma^0_{ij} = \mathcal{P}(g^{-1}, \partial g)$, and solving for ϕ_0, we get the result. □

Lemma 4.3. *At the boundary,*

$$\tilde{\Delta}\phi|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h).$$

Proof. For this formula, we consider how scalar curvature changes under the given conformal change:

$$S = \phi^{-2}\tilde{S} + \phi^{-3}(2-2n)\tilde{\Delta}\phi - \phi^{-4}(n-1)(n-4)|d\phi|^2_{\tilde{g}}.$$

Solving for $\tilde{\Delta}\phi$, we have

$$\tilde{\Delta}\phi = \frac{1}{2(n-1)}(\phi\tilde{S} - \phi^3 S - (n-4)(n-1)\phi^{-1}|d\phi|^2_{\tilde{g}}). \tag{23}$$

Restricting to the boundary, we apply Lemma 3.5 to deal with $\tilde{S}$. This introduces first derivatives of $\tilde{g}$ which, as mentioned earlier, can be written in terms of first derivatives of g and first derivatives of ϕ. Then, we apply Lemma 4.2 to deal with the first derivatives of ϕ, and we replace $\tilde{g}$ with g since they are equal on D. Finally, S is constant, so the result follows. □

To derive the boundary equations of order 2 and (19), we will look at how various components of the Ricci tensor change under the given conformal change. For general components, we have

$$\begin{aligned} Ric_{\alpha\beta} \quad &= \quad \tilde{R}ic_{\alpha\beta} + \phi^{-1}\Big((2-n)\tilde{\nabla}_{\alpha}\tilde{\nabla}_{\beta}\phi - \tilde{\Delta}\phi\,\tilde{g}_{\alpha\beta}\Big) \\ &\qquad + \phi^{-2}\Big((3-n)|d\phi|^2_{\tilde{g}}\,\tilde{g}_{\alpha\beta} + 2(n-2)\phi_{,\alpha}\,\phi_{,\beta}\Big). \end{aligned} \tag{24}$$

With this and the lemmas above, we are ready to derive the boundary equations of order 2 and (19).

Proof of the Boundary Equations of Order 2. When restricted to D the tangential components of (24) reduce to

$$Ric_{ij} = \tilde{R}ic_{ij} + (2-n)\tilde{\nabla}_i\tilde{\nabla}_j\phi - \tilde{\Delta}\phi\tilde{g}_{ij} + (3-n)|d\phi|^2_{\tilde{g}}\tilde{g}_{ij} + 2(n-2)\phi_{,i}\,\phi_{,j}\,. \tag{25}$$

Now, making a number of substitutions, we will arrive at (16). For the left hand side, we again use (1). For the first term on the right hand side, we use Lemma 3.5. Then, as in Lemma 4.3, we replace the first derivatives of $\tilde{g}$ with first derivatives of g and first derivatives of ϕ, and then we use Lemma 4.2 to handle the first derivatives of ϕ.

For the second term, since $\tilde{A}$ is zero, we also have $\tilde{\Gamma}^0_{ij} = (\tilde{g}^{00})^{\frac{1}{2}}\tilde{A}_{ij} = 0$. Hence, since the tangential derivatives of ϕ are zero on D, we have

$$\tilde{\nabla}_i\tilde{\nabla}_j\phi = \phi_{,ij} - \phi_{,\alpha}\,\tilde{\Gamma}^{\alpha}_{ij} = 0.$$

For the third term, we use the formula from Lemma 4.3, and then substitute as above to eliminate the first derivatives of $\tilde{g}$.

The fourth term, which involves only $\tilde{g}$ and first derivatives of ϕ, is taken care of by making the same substitutions as were made earlier. Finally, the fifth term vanishes. Putting all this into (25), and rearranging, we have (16). □

Proof of Equation (19). Differentiating (22) and using (21), we have

$$\partial_k A_{ij} = -\phi_{,0k}\,(g^{00})^{\frac{1}{2}} g_{ij} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g). \tag{26}$$

To gain information about $\phi_{,0k}$, we set $\alpha = k$ and $\beta = 0$ in (24) and solve for $\tilde{\nabla}_k\tilde{\nabla}_0\phi$ while using various results on the other terms in (24). In particular, we use (1) for the left hand side, Lemma 3.5 for $\tilde{Ric}$, Lemma 4.3 for the Laplacian of ϕ, and Lemma 4.2 for the first derivatives of ϕ to end up with

$$\tilde{\nabla}_k\tilde{\nabla}_0\phi = \frac{-1}{2(2-n)}\mathcal{L}g_{k0} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h).$$

On the other hand, $\tilde{\nabla}_k\tilde{\nabla}_0\phi = \phi_{,0k} - \phi_{,\alpha}\,\tilde{\Gamma}^{\alpha}_{0k}$ and we can handle the first derivatives of ϕ and the Christoffel symbol using Lemma 4.2 and the fact that $\partial\tilde{g} = \mathcal{P}(\phi^{-1}, \partial\phi, \partial g)$. Substituting this and rearranging, we get

$$\phi_{,0k} = \frac{-1}{2(2-n)}\mathcal{L}g_{k0} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h).$$

Finally, substituting this into (26) we get (19). ∎

4.3.2 Boundary Equations of Order $2l$, $2 \le l \le (n/2) - 1$

Generating the boundary equations for higher even orders follows the same basic idea as for the boundary equations of order 2. Here, looking at how the lth power of the Laplacian of Ricci changes under a conformal change of the metric gives us an expression for $\mathcal{L}_l$ applied to components of the metric. However it also introduces derivatives of the conformal factor, so to start we will determine how to keep track of derivatives of the conformal factor inductively in order to get the result. In the process, we will see why we do not need to distinguish between tangential and non-tangential components for these equations.

As before, let $\tilde{g}$ be an almost geodesic compactification of order $n-2$ for g_+ with $\iota^*\tilde{g} = \iota^* g = h$ so that $g = \phi^2\tilde{g}$ with $\phi|_D = 1$. We begin with a formula for derivatives of ϕ that we will use a number of times.

Lemma 4.4. *On* U*,*

$$\tilde{\nabla}^p\tilde{\Delta}\phi = \mathcal{P}(\tilde{g}^{-1}, \partial^p\tilde{g}, \phi^{-1}, \partial^{p+1}\phi, \tilde{\nabla}^p\tilde{S}). \tag{27}$$

Proof. Taking covariant derivatives of (23) with respect to $\tilde{g}$ and using the fact that S is constant, we have the result. ∎

The first application of this lemma is the following:

Lemma 4.5. *For* $0 \le p \le n-4$*,*

$$(\partial_0^{p+2}\phi)|_D = -\frac{1}{g^{00}}(2g^{i0}\partial_0^{p+1}\partial_i\phi + g^{ij}\partial_0^p\partial_i\partial_j\phi) + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{p+1}g, \partial^{p+1}\phi, \partial_t^{p+2}h). \tag{28}$$

Proof. Writing $\tilde{\nabla}^p\tilde{\Delta}\phi$ out directly, we have

$$\tilde{\nabla}_{\mu_1}\cdots\tilde{\nabla}_{\mu_p}\tilde{\Delta}\phi = \tilde{g}^{00}\phi_{,00\mu_1\cdots\mu_p} + 2\tilde{g}^{i0}\phi_{,i0\mu_1\cdots\mu_p} + \tilde{g}^{ij}\phi_{,ij\mu_1\cdots\mu_p} + \mathcal{P}(\tilde{g}^{-1}, \partial^{p+1}\phi, \partial^{p+1}\tilde{g}).$$

Applying (27) to the left hand side and then rearranging, we get

$$\phi_{,00\mu_1\cdots\mu_p} = -\frac{1}{\tilde{g}^{00}}(2\tilde{g}^{i0}\phi_{,i0\mu_1\cdots\mu_p} + \tilde{g}^{ij}\phi_{,ij\mu_1\cdots\mu_p}) + \mathcal{P}(\tilde{g}^{-1}, (\tilde{g}^{00})^{-1}, \partial^{p+1}\tilde{g}, \phi^{-1}, \partial^{p+1}\phi, \tilde{\nabla}^p\tilde{S}).$$

From here, we express objects associated with $\tilde{g}$ in terms of objects associated with g by using the fact that $\partial^m\tilde{g} = \mathcal{P}(\phi^{-1}, \partial^m\phi, \partial^m g)$. Also, for $p \le n-4$, we may use Lemma 3.5 for the covariant derivatives of $\tilde{S}$. Making these substitutions we conclude

$$\phi_{,00\mu_1\cdots\mu_p}|_D = -\frac{1}{g^{00}}(2g^{i0}\phi_{,i0\mu_1\cdots\mu_p} + g^{ij}\phi_{,ij\mu_1\cdots\mu_p}) + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{p+1}g, \partial^{p+1}\phi, \partial_t^{p+2}h).$$

In particular, this is true when all $\mu_i = 0$. ∎

We can use this result together with an inductive argument to say even more.

Lemma 4.6. *For $0 \le q \le n-2$ and $0 \le s \le n-2-q$,*

$$(\partial_t^s \partial_0^q \phi)|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{q+s} g, \partial_t^{q+s} h). \tag{29}$$

Proof. If it is already known for some q that $(\partial_0^q \phi)|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^q g, \partial_t^q h)$, then differentiating tangentially proves (29) for $0 \le s \le n-2-q$. Hence it is enough to show the result when $s = 0$. For this, we use induction on q. First, we have some base cases. For $q = 0$, we know $\phi|_{\partial M} = 1$, while for $q = 1$, (21) says $\phi_{,0}|_D = \mathcal{P}(g^{-1}, (g^{00})^{-1}, \partial g)$.

Now let $q \ge 2$, suppose this lemma is true for all $m < q$. Letting $q = p + 2$, we find that every derivative of ϕ on the right hand side of (28) is taken care of by the induction hypothesis. □

If we know that two of the derivatives on ϕ are actually coming from the Laplacian, we can do a little bit better.

Lemma 4.7. *For $0 \le p \le n-4$,*

$$(\tilde{\nabla}^p \tilde{\Delta} \phi)|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{p+1} g, \partial_t^{p+2} h). \tag{30}$$

Proof. By (27), $\tilde{\nabla}^p \tilde{\Delta}\phi = \mathcal{P}(\tilde{g}^{-1}, \partial^p \tilde{g}, \phi^{-1}, \partial^{p+1}\phi, \tilde{\nabla}^p \tilde{S})$, and as before, we use the fact that $\partial^m \tilde{g} = \mathcal{P}(\phi^{-1}, \partial^m \phi, \partial^m g)$ to eliminate derivatives of $\tilde{g}$. Then Lemma 4.6 tells us how to deal with the derivatives of ϕ, and Lemma 3.5 tells us how to deal with the derivatives of $\tilde{S}$. Inserting these equations into (27) produces the result. □

The next lemma is important for showing that we will not need generalizations of the Neumann data for our system.

Lemma 4.8. *For $0 \le p \le n-4$,*

$$(\tilde{\nabla}^{p-2} \tilde{\Delta} \tilde{\nabla}^2 \phi)|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{p+1} g, \partial_t^{p+2} h). \tag{31}$$

Proof. First note that by rearranging covariant derivatives, we have

$$\tilde{\Delta} \tilde{\nabla}^2 \phi = \tilde{\nabla}^2 \tilde{\Delta} \phi + \mathcal{P}(\tilde{g}^{-1}, \partial^3 \tilde{g}, \partial^3 \phi).$$

Hence,

$$\tilde{\nabla}^{p-2} \tilde{\Delta} \tilde{\nabla}^2 \phi = \tilde{\nabla}^p \tilde{\Delta} \phi + \mathcal{P}(\tilde{g}^{-1}, \partial^{p+1} \tilde{g}, \partial^{p+1} \phi).$$

Once again, we have $\partial^m \tilde{g} = \mathcal{P}(\phi^{-1}, \partial^m \phi, \partial^m g)$. When we restrict to the boundary, Lemma 4.6 and Lemma 4.7 allow us to replace all derivatives of ϕ and this gives us the result. □

With the derivatives of ϕ understood, we are ready to finish the proof of Proposition 4.1 by deriving the boundary equations of order $2l$.

Proof of the Boundary Equations of Order $2l$, $2 \le l \le (n/2) - 1$. To start, we look at how the covariant derivatives of the Ricci tensor are affected by a conformal change. For $m = l - 1$, the conformal change formula for covariant derivatives gives us

$$\nabla_{\mu_1} \cdots \nabla_{\mu_{2m}} Ric_{\alpha\beta} = \tilde{\nabla}_{\mu_1} \cdots \tilde{\nabla}_{\mu_{2m}} Ric_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial^{2m} g, \phi^{-1}, \partial^{2m} \phi, \partial^{2m-1} Ric).$$

Next, we use the conformal change formula for the Ricci tensor to get

$$\begin{aligned}
\nabla_{\mu_1} \cdots \nabla_{\mu_{2m}} Ric_{\alpha\beta} \quad = \quad & \tilde{\nabla}_{\mu_1} \cdots \tilde{\nabla}_{\mu_{2m}} \tilde{Ric}_{\alpha\beta} \\
& + \phi^{-1} \Big((2-n) \tilde{\nabla}_{\mu_1} \cdots \tilde{\nabla}_{\mu_{2m}} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \phi - \tilde{\nabla}_{\mu_1} \cdots \tilde{\nabla}_{\mu_{2m}} \tilde{\Delta} \phi \, \tilde{g}_{\alpha\beta} \Big) \\
& + \mathcal{P}(g^{-1}, \partial^{2m+1} g, \phi^{-1}, \partial^{2m+1} \phi).
\end{aligned}$$

Note that the polynomial representing lower order terms loses its dependence on Ric, with the slack being picked up by the extra derivative on g. We also pick up an extra derivative of ϕ from the conformal change of the Ricci tensor.

Now, we trace with respect to $g = \phi^2 \tilde{g}$ to get

$$\begin{aligned}\Delta^m Ric_{\alpha\beta} &= \phi^{-2m}\tilde{\Delta}^m \tilde{R}ic_{\alpha\beta} \\ &\quad +\phi^{-2m-1}\Big((2-n)\tilde{\Delta}^m\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\phi - \tilde{\Delta}^m\tilde{\Delta}\phi\,\tilde{g}_{\alpha\beta}\Big) \\ &\quad +\mathcal{P}(g^{-1},\partial^{2m+1}g,\phi^{-1},\partial^{2m+1}\phi).\end{aligned}$$

Restricting to the boundary, $\phi = 1$ and we may apply all our lemmas to various terms on the right hand side. In particular, we use Lemma 3.5 for the first term, Lemma 4.8 for the second term, Lemma 4.7 for the third term, and Lemma 4.6 for the terms in the polynomial to conclude

$$\Delta^m Ric_{\alpha\beta} = \mathcal{P}(g^{-1},(g^{00})^{-\frac{1}{2}},\partial^{2m+1}g,\partial_t^{2m+2}h).$$

Note that we cannot draw this conclusion when $m = 0$, because then there are no Laplacians to rearrange on the Hessian of ϕ in the second term, and so we cannot make use of Lemma 4.8. This is why we need the boundary equations of order 3.

The last step is to use (1) on the left hand side to get (18). □

5 Local and Global Regularity

Using the boundary value problem that was derived in the last section, we are ready to prove Theorems A and B, which we restate here.

Theorem A. *Let $\overline{M}$ be an n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. Let $p \in \partial M$ and let $U \subset \overline{M}$ be a neighborhood of p with boundary portion $D = U \cap \partial M$. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on $U \cap M$. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(D)$, where $k \geq r$ and $0 < \gamma < 1$. Given a C^∞ coordinate system on the boundary in a neighborhood of p, there is a coordinate system on a neighborhood $V \subset U$ of p that is $C^{r,\sigma}$ compatible with the given smooth structure and which restricts to the given coordinates on the boundary, and there is a defining function $\rho \in C^{r-1,\sigma}(V)$ in the new coordinates, such that $\rho^2 g_+$ has boundary metric h and in the new coordinates, $\rho^2 g_+ \in C^{k,\gamma}(V)$.*

Theorem B. *Let $\overline{M}$ be a compact n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on M. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(\partial M)$, where $k \geq r$ and $0 < \gamma < 1$. Then there is a $C^{r,\sigma}$ diffeomorphism $\Psi : \overline{M} \longrightarrow \overline{M}$ which restricts to the identity on the boundary and there is a defining function $\check{\rho} \in C^{r-1,\sigma}(\overline{M})$ such that $\check{\rho}^2\Psi^*(g_+)$ has boundary metric h and $\check{\rho}^2\Psi^*(g_+) \in C^{k,\gamma}(\overline{M})$.*

To prove Theorem A, we apply classical elliptic regularity results to the boundary value problem that was derived in the previous section. We use a bootstrap argument, each step of which requires its own inductive argument. Theorem B follows from Theorem A by studying the regularity of the atlas of harmonic coordinates that we use and then applying an approximation theorem.

5.1 Local Boundary Regularity

Before proving Theorem A, we have two supporting lemmas. The first lemma shows that the various types of boundary conditions all produce the same regularity result, and is basically an application of classical second order elliptic boundary regularity results. In order to apply these results, we observe that when composing $\mathcal{L}$ with $\mathcal{L}_l$, we have

$$\mathcal{L}(\mathcal{L}_l g_{\alpha\beta}) = \mathcal{L}_{l+1}g_{\alpha\beta} + \mathcal{P}(g^{-1},\partial^{2l+1}g),$$

where the polynomial of lower order terms is zero if $l = 0$. Hence, we can think of $\mathcal{L}_{l+1}g_{\alpha\beta}$ as a second order operator acting on $\mathcal{L}_l g_{\alpha\beta}$, modulo lower order terms.

Lemma 5.1. *Let $U \subset \overline{M}$ be a boundary adapted coordinate domain with boundary portion D, let g be a metric on U, and let h be a metric on D. Let $p \geq 0$, $0 \leq l \leq (n/2) - 1$, and $0 < \sigma < 1$. Suppose $g_{\alpha\beta} \in C^{p+2l+1,\sigma}(U) \cap C^{2l+2}(U \cap M)$ and $h_{ij} \in C^{p+2l+2,\sigma}(D)$. Also, suppose $\mathcal{L}_{l+1}g_{\alpha\beta} \in C^{p,\sigma}(U)$.*

Moreover, referring to (14)–(18) *and* (19) *of Proposition 4.1, suppose the components of g and h solve the boundary equations of order 0 and 1 and* (19) *if $l = 0$, order 2 and 3 if $l = 1$, and order $2l$ if $l \geq 2$. Then $\mathcal{L}_l g_{\alpha\beta} \in C^{p+2,\sigma}(U)$.*

Proof. Note that we may use regularity results for linear equations since we are working with a fixed metric. With this in mind, we consider three cases.

Case 1: $2 \leq l \leq (n/2) - 1$. By our observation about composition of $\mathcal{L}$ with $\mathcal{L}_l$, we have

$$\mathcal{L}(\mathcal{L}_l g_{\alpha\beta}) = \mathcal{L}_{l+1} g_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial^{2l+1} g) \tag{32}$$

on $U \cap M$. Also, as given by (18), the boundary equation of order $2l$ is

$$(\mathcal{L}_l g_{\alpha\beta})|_D = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^{2l-1} g, \partial_t^{2l} h) \tag{33}$$

We view (32) as a linear scalar equation for $\mathcal{L}_l g_{\alpha\beta}$ and (33) as a Dirichlet boundary condition. By the regularity hypotheses provided, we have $\mathcal{L}_l g_{\alpha\beta} \in C^{p+1,\sigma}(U) \cap C^2(U \cap M) \subset C^0(U) \cap C^2(U \cap M)$. Moreover, the right hand side of (32) is in $C^{p,\sigma}(U)$, and the right hand side of (33) is in $C^{p+2,\sigma}(D)$. Hence, by local regularity results for the Dirichlet problem [GT01], we may conclude that $\mathcal{L}_l g_{\alpha\beta} \in C^{p+2,\sigma}(U)$.

Case 2: $l = 1$. In this case, similar to (32), we have

$$\mathcal{L}(\mathcal{L} g_{\alpha\beta}) = \mathcal{L}_2 g_{\alpha\beta} + \mathcal{P}(g^{-1}, \partial^3 g) \tag{34}$$

on $U \cap M$, but now our boundary conditions require more care than the previous argument. We proceed in three steps. In the first step, we focus only on tangential components of g. For these, the boundary equations of order 2 provide Dirichlet conditions and combined with (34), we conclude that $\mathcal{L} g_{ij} \in C^{p+2,\sigma}(U)$ just as in case 1.

For the next two steps, we will use the boundary equations of order 3 by isolating the terms with $\beta = 0$ in (17). Writing this equation out and dividing everything by $(g^{00})^{\frac{1}{2}}$, we have

$$\mathcal{N}(\mathcal{L} g_{\alpha 0}) = -(g^{00})^{-\frac{1}{2}} g^{\eta i} \partial_\eta (\mathcal{L} g_{\alpha i}) + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial^2 g). \tag{35}$$

For the second step, we let $\alpha = j$. Then the first term on the right hand side of (35) is in $C^{p+1,\sigma}(D)$ as a result of the first step above. The second term on the right is also in $C^{p+1,\sigma}(D)$. Moreover, $\mathcal{L} g_{\alpha\beta} \in C^1(U) \cap C^2(U \cap M)$ and the right hand side of (34) is in $C^{p,\sigma}(U)$. Hence, treating the pair (34), (35) as a Neumann problem for a linear scalar equation, we may use elliptic regularity [Mir70] to conclude $\mathcal{L} g_{0j} \in C^{p+2,\sigma}(U)$.

For the third step we let $\alpha = 0$ in (35). Then, by the second step, the first term on the right hand side is in $C^{p+1,\sigma}(D)$, as is the second term. Therefore, by the same argument as in the second step, we have $\mathcal{L} g_{00} \in C^{p+2,\sigma}(U)$. These three steps together handle every component of $\mathcal{L} g$.

Case 3: $l = 0$. This case is similar to case 2 in that we deal with the tangential and nontangential components of g separately. For any components we have

$$\mathcal{L} g_{\alpha\beta} \in C^{p,\sigma}(U), \tag{36}$$

which is provided for us by the hypotheses. As in case 2, we work in three steps, although the second and third steps are reversed relative to case 2. For the first step, we focus on the tangential components and the argument is essentially the same as for the tangential components in case 2. We are given that $g_{ij} \in C^0(U) \cap C^2(U \cap M)$ and the boundary equations of order 0 are just the statements that $g_{ij} = h_{ij}$, which are given to be in $C^{p+2,\sigma}(D)$. Hence, by boundary regularity results for the Dirichlet problem, we have $g_{ij} \in C^{p+2,\sigma}(U)$.

For the second and third steps, we will make use of the boundary equations of order 1, but first an analysis of (19) will provide an essential simplification. To start, we note that $\mathcal{L} g_{k0}$, the metric g and all first derivatives of g, and up to second derivatives of h are in $C^{p,\sigma}(U)$. Hence (19) tells us that $\partial_k A_{ij} \in C^{p,\sigma}(D)$ for $1 \leq k \leq n-1$ and so $A_{ij} \in C^{p+1,\sigma}(D)$. Since we are in boundary adapted coordinates,

$$\begin{aligned} A_{ij} &= (g^{00})^{-\frac{1}{2}} \Gamma_{ij}^0 \\ &= \frac{1}{2} (g^{00})^{-\frac{1}{2}} g^{0\alpha} (\partial_j g_{i\alpha} + \partial_i g_{j\alpha} - \partial_\alpha g_{ij}). \end{aligned} \tag{37}$$

From step 1, all tangential components of g are in $C^{p+2,\sigma}(U)$. With this and the regularity determined for A_{ij}, we rearrange (37) and conclude

$$g_{i0},_j + g_{j0},_i \in C^{p+1,\sigma}(D). \tag{38}$$

Multiplying by g^{ij} and summing, we also have

$$g^{ij} g_{i0},_j \in C^{p+1,\sigma}(D). \tag{39}$$

With these facts, we focus on the boundary equations of order 1. For the second step, we look at (15) with $\alpha = 0$. Writing this out by separating terms with 0 as an index, we have

$$g^{ij}\partial_i g_{0j} + g^{0j}\partial_0 g_{0j} + g^{j0}\partial_j g_{00} + g^{00}\partial_0 g_{00} - \frac{1}{2}(g^{ij}\partial_0 g_{ij} + 2g^{0j}\partial_0 g_{0j} + g^{00}\partial_0 g_{00}) = 0.$$

The second term cancels with the middle term in parentheses and the fourth term partially cancels the last term in parentheses. Rearranging then gives

$$(g^{j0}\partial_j + \frac{1}{2} g^{00}\partial_0) g_{00} = \frac{1}{2} g^{ij}\partial_0 g_{ij} - g^{ij}\partial_i g_{0j}.$$

By (39) and the fact that $g_{ij} \in C^{p+2,\sigma}(U)$, we have

$$(g^{j0}\partial_j + \frac{1}{2} g^{00}\partial_0) g_{00} \in C^{p+1,\sigma}(D). \tag{40}$$

Combined with (36), this gives us a regular oblique derivative problem. Since $g \in C^1(U) \cap C^2(U \cap M)$, we apply boundary regularity for such a problem [Mir70] to conclude $g_{00} \in C^{p+2,\sigma}(U)$.

For the third step, we repeat this analysis using (15) with $\alpha = j$. In this case we have

$$g^{\eta k}\partial_\eta g_{jk} + g^{i0}\partial_i g_{j0} + g^{00}\partial_0 g_{j0} - \frac{1}{2}(g^{ik}\partial_j g_{ik} + 2g^{i0}\partial_j g_{i0} + g^{00}\partial_j g_{00}) = 0. \tag{41}$$

The first term along with the first and last terms in parentheses are in $C^{p+1,\sigma}(U)$ by the first and second steps. Moreover, by (38) we may replace $g^{i0}\partial_j g_{i0}$ by $-g^{i0}\partial_i g_{j0}$ modulo a term in $C^{p+1,\sigma}$. Using these facts, (41) simplifies to

$$(g^{i0}\partial_i + \frac{1}{2} g^{00}\partial_0) g_{j0} \in C^{p+1,\sigma}(D). \tag{42}$$

Therefore, by the same argument as in the second step, $g_{j0} \in C^{p+2,\sigma}(U)$. These three steps together handle every component of g, and the lemma is proved. □

Our second lemma uses Lemma 5.1 to provide us with each step in the eventual bootstrap.

Lemma 5.2. *Let $U \subset \overline{M}$ be a boundary adapted coordinate domain with boundary portion D. Let $m \geq n-1$, let g be a metric in $C^{m,\sigma}(U) \cap C^n(U \cap M)$, and let h be a metric in $C^{m+1,\sigma}(D)$. Suppose the components of g and h solve the boundary value problem* (13)–(18) *and* (19). *Then the components of g are in $C^{m+1,\sigma}(U)$.*

Proof. We prove this using a reverse induction argument on the power of $\mathcal{L}$ acting on components of g. For the base case, observe that the second term on the left hand side of (13) is in $C^{m-n+1,\sigma}(U)$, so setting $l = (n/2) - 1$ and $p = m - n + 1$, we find that the conditions of Lemma 5.1 are satisfied. Hence we may conclude that $\mathcal{L}_{\frac{n}{2}-1} g_{\alpha\beta} \in C^{m-n+3,\sigma}(U)$.

For the induction step, suppose that for some l with $0 \leq l \leq (n/2) - 1$, we have $\mathcal{L}_{l+1} g_{\alpha\beta} \in C^{m-2l-1,\sigma}(U)$. Setting $p = m - 2l - 1$, we find that the conditions of Lemma 5.1 are satisfied, and so $\mathcal{L}_l g_{\alpha\beta} \in C^{m-2l+1,\sigma}(U)$.

The induction terminates at $l = 0$, in which case $p = m - 1$ and we are left with $g_{\alpha\beta} \in C^{m+1,\sigma}(U)$, which was our goal. □

Proof of Theorem A. By Proposition 2.3, g_+ has a constant scalar curvature compactification $g = \rho^2 g_+$ in $C^{r-1,\sigma}$ near p with boundary metric h, where for any smooth defining function $\hat{\rho}$, we have $\rho/\hat{\rho} \in C^{r-1,\sigma}$ near p. By Proposition 2.1 there are harmonic coordinates for g in a neighborhood V about p which are $C^{r,\sigma}$ compatible with the given smooth structure on U and which restrict to coordinates in the given smooth structure on D. Note that in these new coordinates, $\rho \in C^{r-1,\sigma}(V)$. With this compactification and these coordinates, the conditions of Proposition 4.1 are satisfied, so g satisfies the resulting boundary value problem. We will apply Lemma 5.2 inductively to conclude the desired regularity.

We consider two cases depending on the relationship between γ and σ. If $\gamma \leq \sigma$, we may replace σ by γ. Then, applying Lemma 5.2 inductively, we have the result.

On the other hand, if $\gamma > \sigma$, then after applying Lemma 5.2 once, we have $g \in C^{r,\sigma} \subset C^{r-1,\gamma}$. We then apply Lemma 5.2 inductively, starting with $g \in C^{r-1,\gamma}$, and the result again follows. □

5.2 Global Regularity

Theorem B now follows from Theorem A, a patching argument, and an approximation theorem. Twice in the patching argument, we will use the general fact, as discussed in Section 2.2, that if a metric g is $C^{k,\gamma}$ with respect to a given coordinate chart, then harmonic coordinates are $C^{k+1,\gamma}$ with respect to the given coordinates.

Proof of Theorem B. First observe that by the remark after Proposition 2.3, we have a global constant scalar curvature compactification g. Letting $\mathcal{A}$ be the maximal C^∞ atlas for $\overline{M}$, we construct a new atlas $\mathcal{B}$, about each point in $\overline{M}$ by choosing harmonic coordinates for g. In particular, for any point in ∂M, we use the coordinates in the proof of Theorem A. Since g is in $C^{r-1,\sigma}$ with respect to $\mathcal{A}$, such coordinates are $C^{r,\sigma}$ compatible with $\mathcal{A}$ since they are harmonic. Moreover, by Theorem A, g is in $C^{k,\gamma}$ in each new coordinate chart, while on the interior, by Proposition 2.4, g is C^∞. Hence, g is in $C^{k,\gamma}$ on all such charts. As a consequence, the collection of these harmonic coordinate charts must be $C^{k+1,\gamma}$ compatible with one another.

Now let $id_1 : (\overline{M}, \mathcal{A}) \longrightarrow (\overline{M}, \mathcal{B})$ be the identity map. Let $id_2 = id_1^{-1}$ and note that $id_2^*(g_+)$ is just g_+ in the new atlas $\mathcal{B}$. By Whitney approximation, id_1, which is $C^{r,\sigma}$, can be approximated in C^1 by a diffeomorphism $\Theta : (\overline{M}, \mathcal{A}) \longrightarrow (\overline{M}, \mathcal{B})$ which is $C^{k+1,\gamma}$ and restricts to the identity on ∂M. With this, let $\Psi = id_2 \circ \Theta$ and $\check{\rho} = \Theta^* \rho$. By Theorem A, $\rho^2 id_2^*(g_+)$ has the desired boundary characteristics. Since $\rho^2 id_2^*(g_+)$ is in $C^{k,\gamma}(\overline{M}, \mathcal{B})$, we have

$$\check{\rho}^2 \Psi^*(g_+) = \Theta^*(\rho^2 id_2^*(g_+)) \in C^{k,\gamma}(\overline{M}, \mathcal{A}).$$

□

I would like to thank Olivier Biquard for the idea of using Whitney approximation to streamline this result.

6 Regularity of the Defining Function

This section culminates in the proof of Theorem C, which we restate here.

Theorem C. *Let $\overline{M}$ be a compact n-dimensional C^∞ manifold with boundary, $n \geq 4$ and even. For $r \geq n$ and $0 < \sigma < 1$, let g_+ be a $C^{r-1,\sigma}$ conformally compact Einstein metric on M. Suppose that the conformal infinity of g_+ contains a metric $h \in C^{k,\gamma}(\partial M)$, where $k \geq r$ and $k \geq n+1$, and $0 < \gamma < 1$. Then there is a $C^{r,\sigma}$ diffeomorphism $\Psi : \overline{M} \longrightarrow \overline{M}$ which restricts to the identity on ∂M such that $\Psi^*(g_+)$ is $C^{k,\gamma'}$ conformally compact for some γ', $0 < \gamma' \leq \gamma$.*

As discussed in the introduction, Theorem B does not result in a $C^{k,\gamma}$ conformally compact metric because while $\rho^2 g_+$ is in $C^{k,\gamma}$, the defining function that is used to generate the compactification need not be $C^{k+1,\gamma}$. Since the regularity for the compactification produced in Theorem B relies on a change of smooth structure, and the defining function used wasn't necessarily $C^{k+1,\gamma}$ up to the boundary in the first place, it is not immediately clear what can be said about the defining function.

We will analyze this problem via the singular Yamabe problem, and we will find that Theorem C follows as a corollary to the results of our analysis.

6.1 The Singular Yamabe Problem

Let $\overline{M}$ be a $C^{k+1,\gamma}$ n-dimensional compact manifold with boundary, where $k \geq 2$ and $0 < \gamma < 1$. Our discussion here does not depend on whether n is even or odd, so until the proof of Theorem C, n need not be even. Let $\mathring{g}$ be a $C^{k,\gamma}$ conformally compact metric, with $k \geq 2$. In the singular Yamabe problem, the goal is to find a function u such that the metric $g_+ = u^{\frac{4}{n-2}}\mathring{g}$ has constant scalar curvature S_+, which we take to be $-n(n-1)$. In this setting, the conformal change formula for scalar curvature becomes

$$\mathring{\Delta} u - \frac{(n-2)}{4(n-1)}\mathring{S}u + \frac{(n-2)}{4(n-1)}S_+ u^{\frac{n+2}{n-2}} = 0.$$

This problem is studied as a special case of equation (7.1.1) in [AC96], and there it is shown that there is a function u that solves this equation, and hence the associated singular Yamabe problem. Moreover, the solution is unique in the class of uniformly bounded, uniformly bounded away from zero, C^2 functions on M. The boundary regularity of u is also studied, and it is this regularity that we will use to prove Theorem C.

6.2 Regularity in General

At this point, we state the regularity result that we will be using. This is a special case of Theorem 7.4.7 in [AC96]. Their result is stated for C^∞ manifolds, but it can be checked that their results are valid for manifolds with lower regularity smooth structures. Also see [ACF92] for a similar result, and [Maz91] for a more general discussion of the singular Yamabe problem.

Theorem 6.1 (Andersson, Chruściel)**.** *Let $\overline{M}$ be an n-dimensional $C^{k+1,\gamma}$ compact manifold with boundary, $k \geq 2$, $0 < \gamma < 1$. Suppose $\mathring{g}$ is $C^{k,\gamma}$ conformally compact and let $u \in C^2(M)$ be a function that is uniformly bounded, uniformly bounded above zero, and that satisfies*

$$\mathring{\Delta} u - \frac{(n-2)}{4(n-1)}\mathring{S}u - \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0.$$

Then for some $0 < \gamma' < 1$,

$$u \in \begin{cases} C^{k,\gamma'}(\overline{M}) & (k \leq n-1) \\ C^{n-1,\gamma'}(\overline{M}) & (k \geq n). \end{cases}$$

Moreover, if $k > n$ then there is a function $\Phi \in C^{k,\gamma'}(\overline{M})$ and a sequence of functions

$$\phi_j \in \bigcap_{i=0}^{n} y^{jn-i}C^{k-n+i,\gamma'}(\overline{M}),\ j = 1,\ldots,N$$

where N is the smallest integer such that $N > k/n$ and y is a $C^{k+1,\gamma}$ defining function for ∂M, such that

$$u = \Phi + \sum_{j=1}^{N}\phi_j \log^j(y) \tag{43}$$

in $\overline{M}$. Finally, if $y^{-n}\phi_1|_{\partial M} = 0$, then all the ϕ_j can be taken to be zero and $u \in C^{k,\gamma'}(\overline{M})$.

We note here that this theorem is global, and currently no local analogue is known. It is also natural to guess that the theorem is also true setting $\gamma' = \gamma$, and that (43) is valid when $k = n$.

6.3 Regularity when g_+ is Einstein

In our setting, the metric g_+ is Einstein, which is stronger than simply having constant scalar curvature. We recall that for us, g_+ being Einstein means that $Ric_+ = -(n-1)g_+$. The following proposition shows that in this case, the ϕ_j are all zero.

Proposition 6.2. *Let $\overline{M}$ be an n-dimensional $C^{k+1,\gamma}$ compact manifold with boundary, where $k \geq 2, k \neq n$, and $0 < \gamma < 1$, and let $\mathring{g}$ be $C^{k,\gamma}$ conformally compact. Suppose $g_+ = u^{\frac{4}{n-2}}\mathring{g}$ is Einstein, where $u \in C^2(M)$ is uniformly bounded and uniformly bounded above zero. Then $u \in C^{k,\gamma'}(\overline{M})$ for some γ', $0 < \gamma' < 1$.*

Proof. Note that for $k < n$, the Einstein condition is not necessary, and the result is immediate from Theorem 6.1 if we only know that $S_+ = -n(n-1)$. For $k > n$, we will use the expression (43) for u in Theorem 6.1 and use the fact that g_+ is Einstein to show that $(y^{-n}\phi_1)|_{\partial M} = 0$.

Working in a boundary adapted coordinate system $\{x^\alpha\}$ with $x^0 = y$, we let $u^{\frac{2}{n-2}} = e^v$ so that $g_+ = e^{2v}\mathring{g}$. To make use of the condition that g_+ is Einstein, consider the way that the Ricci tensor changes under conformal change of the metric:

$$\mathring{R}ic_{\alpha\beta} = (\mathring{\Delta}v)\mathring{g}_{\alpha\beta} + (n-2)(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta v + |dv|^2_{\mathring{g}}\mathring{g}_{\alpha\beta} - v_\alpha v_\beta) + (Ric_+)_{\alpha\beta}.$$

For our purposes, it will be enough to consider the trace free part of $\mathring{R}ic$, and we note that since g_+ is Einstein, the trace free part of the last term above is zero. We denote the trace free part of a tensor T by $\mathrm{tf}(T)$, so in our case we have

$$\mathrm{tf}(\mathring{R}ic_{\alpha\beta}) = (n-2)\,\mathrm{tf}(\mathring{\nabla}_\alpha\mathring{\nabla}_\beta v - v_\alpha v_\beta).$$

Our analysis will actually apply to $y\,\mathrm{tf}(\mathring{R}ic_{\alpha\beta})$, so we focus on the equation

$$y\,\mathrm{tf}(\mathring{R}ic_{\alpha\beta}) = (n-2)\,\mathrm{tf}(y\,\mathring{\nabla}_\alpha\mathring{\nabla}_\beta v - y\,v_\alpha v_\beta). \tag{44}$$

Focusing first on the left hand side of this equation, we determine how $\mathring{R}ic$ behaves at the boundary by making another conformal change, namely $g = y^2\mathring{g}$. We have

$$\mathring{R}ic_{\alpha\beta} = Ric_{\alpha\beta} + y^{-1}\Big((n-2)\nabla_\alpha\nabla_\beta y + \Delta y\, g_{\alpha\beta}\Big) + y^{-2}\Big((1-n)|dy|^2_g g_{\alpha\beta}\Big).$$

Hence

$$y\,\mathrm{tf}(\mathring{R}ic_{\alpha\beta}) = \mathrm{tf}(yRic_{\alpha\beta} + (n-2)\nabla_\alpha\nabla_\beta y)$$

which is $C^{k-2,\gamma}(\overline{M})$, since g is in $C^{k,\gamma}(\overline{M})$. Hence $y\,\mathrm{tf}(\mathring{R}ic_{\alpha\beta}) \in C^{n-1,\gamma}(\overline{M})$, since $k > n$.

For the right hand side of (44), we use (43) to better understand the derivatives of v. We start by making a few reductions with regard to (43). First, by making a conformal change to absorb Φ into $\mathring{g}$, we may say $\Phi = 1$. Also, we have $\phi_1 = (n-2)y^n\psi/2$, where $\psi \in \bigcap_{i=0}^n y^{-i}C^{k-n+i,\gamma'}(\overline{M})$. Finally, let $w = \sum_{j=1}^N \phi_j \log^j(y)$. We can then write v as follows:

$$\begin{aligned} v &= \frac{2}{n-2}(w + \log(1+w) - w) \\ &= y^n\log(y)\psi + \frac{2}{n-2}\left(\sum_{j=2}^N \phi_j\log^j(y) + \log(1+w) - w\right) \\ &= y^n\log(y)\psi + f, \end{aligned}$$

where $f = \frac{2}{n-2}\Big(\sum_{j=2}^N \phi_j\log^j(y) + \log(1+w) - w\Big)$. By Theorem 6.1, we know that $u \in C^{n-1}(\overline{M})$ and $\phi_j \in \bigcap_{i=0}^n y^{jn-i}C^{k-n+i,\gamma'}(\overline{M})$. From these facts we find that $f \in o(y^n) \cap C^{n-1,\gamma'}(\overline{M})$. Note that this implies $\partial^l f \in o(y^{n-l})$ for $l \leq n-1$.

Differentiating and using the characterization of ψ above, we have

$$v_{,\alpha} = ny^{n-1}y_{,\alpha}\,\log(y)\psi + O(y^{n-1}) \tag{45}$$

and

$$v_{,\alpha\beta} = n(n-1)y^{n-2}y_{,\alpha}\,y_{,\beta}\,\log(y)\psi + O(y^{n-2}). \tag{46}$$

Our next step is to write out the second covariant derivative of v with respect to $\mathring{g}$ in terms of $g = y^2\mathring{g}$. In doing so, we use the following transformation rule for the Christoffel symbol:

$$\mathring{\Gamma}^\eta_{\alpha\beta} = \Gamma^\eta_{\alpha\beta} - y^{-1}(y_{,\alpha}\,\delta^\eta_\beta + y_{,\beta}\,\delta^\eta_\alpha - y_{,\mu}\,g^{\eta\mu}g_{\alpha\beta}).$$

With this, (45), and (46), we have

$$\begin{aligned} \mathring{\nabla}_\alpha\mathring{\nabla}_\beta v &= (v_{,\alpha\beta} - v_{,\eta}\,\mathring{\Gamma}^\eta_{\alpha\beta}) \\ &= y^{n-2}\log(y)\psi\Big((n^2+n)y_{,\alpha}\,y_{,\beta} - |dy|^2_g g_{\alpha\beta}\Big) + O(y^{n-2}). \end{aligned}$$

Since we will be inserting this into (44), we multiply this equation by y, and observe that the trace free part of the second term is zero. Also, we write $y_{,\alpha}\, y_{,\beta} = \delta^0_\alpha \delta^0_\beta$, and note that $\operatorname{tf}(\delta^0_\alpha \delta^0_\beta)$ is not zero because as a map from $T\overline{M}$ to $T^*\overline{M}$, the transformation $\delta^0_\alpha \delta^0_\beta$ is rank one and so cannot be a multiple of the metric. Finally, $v_\alpha v_\beta \in O(y^{n-2})$, so (44) reduces to

$$y \operatorname{tf}(\mathring{R}ic_{\alpha\beta}) = (n-2)(n^2+n) y^{n-1} \psi \log(y) \operatorname{tf}(\delta^0_\alpha \delta^0_\beta) + O(y^{n-1}).$$

Combining this with our analysis of the left hand side of (44), we conclude

$$(n-2)(n^2+n) y^{n-1} \psi \log(y) \operatorname{tf}(\delta^0_\alpha \delta^0_\beta) + O(y^{n-1}) \in C^{n-1,\gamma}(\overline{M}).$$

This requires that $\psi = 0$ at every point in ∂M, and so by Theorem 6.1, $u \in C^{k,\gamma'}(\overline{M})$ □

6.4 Regularity of the Defining Function

Using Proposition 6.2, we have the following proposition.

Proposition 6.3. *Let $\overline{M}$ be an n-dimensional $C^{k+1,\gamma}$ compact manifold with boundary, $k \neq n$, with $C^{k+1,\gamma}$ defining function y. Suppose g_+ is an Einstein metric on M with the property that $g = \rho^2 g_+$ extends to a $C^{k,\gamma}$ metric on $\overline{M}$, where $\rho \in C^2(M) \cap C^1(\overline{M})$ is a defining function for ∂M. Then $\rho/y \in C^{k,\gamma'}(\overline{M})$, for some γ', $0 < \gamma' < \gamma$, and therefore g_+ is $C^{k,\gamma'}$ conformally compact.*

Proof. We need to show that $y^2 g_+$ extends to a $C^{k,\gamma'}$ metric on $\overline{M}$. We will use Proposition 6.2 and some manipulation of the defining functions to achieve the result. Observe that $g_+ = u^{\frac{4}{n-2}} \mathring{g}$, where $u = (\rho/y)^{\frac{2-n}{2}}$ and that $\mathring{g} = y^{-2} g$ is $C^{k,\gamma}$ conformally compact. Moreover, u satisfies all the hypotheses in Proposition 6.2, and so $\rho/y \in C^{k,\gamma'}(\overline{M})$. Now we observe that

$$y^2 g_+ = y^2 \rho^{-2} \rho^2 g_+ = u^{\frac{4}{n-2}} g,$$

and both u and g are $C^{k,\gamma'}$ on $\overline{M}$. □

Now we prove Theorem C as a corollary.

Proof of Theorem C. The proof of Theorem B provides us with a $C^{k+1,\gamma}$ atlas $\mathcal{B}$ which is $C^{r,\sigma}$ related to $\mathcal{A}$, a $C^{k+1,\gamma}$ diffeomorphism $\Theta : (\overline{M}, \mathcal{A}) \longrightarrow (\overline{M}, \mathcal{B})$, and a defining function $\rho \in C^{r-1,\sigma}(\overline{M}, \mathcal{B})$. In this setting, by Proposition 6.3, there is a defining function $y \in C^{k+1,\gamma}(\overline{M}, \mathcal{B})$ such that $y^2 id_2^*(g_+) \in C^{k,\gamma'}(\overline{M}, \mathcal{B})$. Similarly to the proof of Theorem B, we find that $\Psi^*(g_+)$ is $C^{k,\gamma'}$ conformally compact. Indeed, $\Theta^*(y) \in C^{k+1,\gamma}(\overline{M}, \mathcal{A})$ and

$$\Theta^*(y)^2 \Psi^*(g_+) = \Theta^*(y^2 id_2^*(g_+)),$$

which is in $C^{k,\gamma'}(\overline{M}, \mathcal{A})$. □

7 An Alternative Approach: Viewing the Boundary Value Problem as a System

The boundary value problem (13)–(18) can naturally be interperated as an elliptic system with accompanying boundary conditions. Here, we introduce the framework for general elliptic boundary value problems following [ADN64] and we show that, by incorporating (19), a mild adjustment of (13)–(18) fits this framework. We finish with a discussion of the applicability of various regularity results for such problems, as provided in [ADN64] and [Mor66].

7.1 General Elliptic Boundary Value Problems

Here, we discuss the framework for general elliptic boundary value problems, following closely the treatment in [ADN64]. We focus on linear systems, but we note that Morrey treats the nonlinear case in [Mor66]. We also restrict our attention to boundary regularity, although [ADN64] and [Mor66] deal with interior regularity as an essential precursor to their boundary regularity results.

Throughout this discussion, Let U be a coordinate domain of a C^∞ manifold with boundary $\overline{M}$, and let $D = U \cap \partial M$ be the boundary portion of U. Let $\{x^\alpha\}$ be boundary adapted coordinates on U.

On U, we consider the system

$$L_{st}(\partial)u^t = F_s, \tag{47}$$

where $L_{st}(\partial)$ are the components of an $N \times N$ matrix of differential operators, u^t are the components of a vector of unknowns, and F_s are inhomogeneous terms.

On D, we consider

$$B_{rt}(\partial)u^t = \phi_r, \tag{48}$$

where $B_{rt}(\partial)$ are the components of an $M \times N$ matrix of differential operators and ϕ_q are inhomogeneous terms. Note that N is determined according to how many unknowns there are, while for now, M is not yet determined.

7.1.1 Weights and Ellipticity

In order to determine whether or not our system is elliptic, and whether or not our boundary data are appropriate, we need to know what part of each differential operator should be counted as its principal part. To this end, we introduce integer weights that we attach to each function u^t, and to each equation in the system and the boundary conditions.

We label the weights for the functions u^t by $w(u^t)$, and for the rows of the system and boundary matrices by $w(L_s)$ and $w(B_r)$ respectively. Letting $\mathrm{ord}(\mathcal{O})$ be the order of the operator $\mathcal{O}$, the goal is to find values for these weights such that $\mathrm{ord}(L_{st}(\partial)) \leq w(L_s) + w(u^t)$ and such that $\mathrm{ord}(B_{rt}(\partial)) \leq w(B_r) + w(u^t)$. Given a collection of weights that satisfy these conditions, we can add any integer to all the $w(u^t)$, while subtracting the same integer from $w(L_s)$ and $w(B_r)$ to find another solution. Hence, to eliminate this freedom, we requre that $w(L_s) \leq 0$, with the largest such weight equal to zero. While other choices could be made in this regard, this choice follows the convention used in both [ADN64] and [Mor66].

With these weights in hand, we define the *principal part* of $L_{st}(\partial)$ or $B_{rt}(\partial)$ to be the term of order exactly $w(L_s) + w(u^t)$ or $w(B_r) + w(u^t)$ respectively. We denote this by $L'_{st}(\partial)$, respectively $B'_{rt}(\partial)$. If a component of either matrix has no term of the given order, then its principal part is 0. We also denote the *principal symbol* by $L'_{st}(\xi)$, respectively $B'_{rt}(\xi)$, where ξ is a (real) covector replacing "∂".

We say the system L is *elliptic* if $\det(L'(\xi)) \neq 0$ for any (real) non-zero covector. Letting $m = \frac{1}{2}\deg(\det L'(\xi))$, we say L is *uniformly elliptic* if there is a positive constant a such that $a^{-1}|\xi|^{2m} \leq |\det(L'(\xi))| \leq a|\xi|^{2m}$.

7.1.2 Boundary Equations and the Complementing Condition

The boundary conditions in a given boundary value problem need to be appropriate for the system in order to make the problem well posed. There are two conditions that must be satisfied in order for this to happen. First, with regard to the size of B, the number of conditions M must be equal to m as defined above. Second, the boundary conditions must satisfy a "complementing condition" depending on an algebraic relationship between the principal symbols of $L(\partial)$ and $B(\partial)$.

At each point $p \in D$, consider the characteristic equation $\det(L'(\xi + \tau\nu)) = 0$ where $\xi \in T^*_p(\partial M)$ is nonzero and ν is the inward pointing unit normal covector. If L is elliptic, then as a polynomial in the complex variable τ, this characteristic equation will have m roots τ_r, $1 \leq r \leq m$, with positive imaginary part. Consider the polynomial

$$M^+(\tau) = \prod_{r=1}^{m}(\tau - \tau_r).$$

We say that $B(\partial)$ *satisfies the complementing condition for* $L(\partial)$ if for each ξ, the rows of $B'(\xi + \tau\nu)\mathrm{adj}L'(\xi + \tau\nu)$ are linearly independent modulo M^+, where $\mathrm{adj}L'$ is the matrix adjoint to L' (not the

conjugate transpose). That is to say, if

$$c^r B'_{rt}(\xi + \tau\nu)(\mathrm{adj}L')^{tq}(\xi + \tau\nu) = P^q(\tau)M^+(\tau),$$

where $c^r \in \mathbb{C}$, and P^q are polynomials in τ, then in fact all the c^r are zero.

Occasionally, in order to check the complementing condition, we can simplify our task by observing some general facts about linear independence when working with polynomials:

Lemma 7.1. *Let $\{v_i\}$ be a set of vectors with components that are polynomials in one complex variable, and let P be a polynomial in one complex variable. Let τ_0 be a common root for all the components of all v_i. Then, one of two situations will occur:*

1. *If τ_0 is a root of P, then $\{v_i\}$ is linearly independent modulo P if and only if $\{v_i/(\tau - \tau_0)\}$ is linearly independent modulo $P/(\tau - \tau_0)$.*
2. *If τ_0 is not a root of P, then $\{v_i\}$ is linearly independent modulo P if and only if $\{v_i/(\tau - \tau_0)\}$ is linearly independent modulo P.*

The proof of this lemma is straightforward.

7.2 Application to the Current Problem

Our goal now is to show that the boundary value problem (13)–(18) is indeed an elliptic system with boundary equations that satisfy the complementing condition. We will find that the system is uniformly elliptic. The boundary equations need to be altered however, since (19) must be incorporated in order that (14) – (18) satisfy the complementing condition.

7.2.1 The System and Uniform Ellipticity

Before looking at the boundary equations, we can show that (13) is uniformly elliptic. Our first step is to do some relabeling. The unknowns in our system are the components of our constant scalar compactification g, so by symmetry we have $N = n(n+1)/2$ unknowns, and for what follows, we choose the upper triangular components as our representatives so that we may order them in a well defined way. To do so, we use a non-standard lexicographic ordering for the components $g_{\alpha\beta}$. It is non-standard in that we take "0" to be larger than other integers so that the components of the metric of the form $g_{0\alpha}$ come last in the list, with g_{00} ending the sequence. We may then denote the functions $g_{\alpha\beta}$ by u^t, $1 \leq t \leq N$.

Focusing on (13), we see that our system is quasi-linear, but for our purposes we may take it to be linear since we already have a solution. With this in mind, we rewrite (13) as follows:

$$\begin{cases} L_{st}u^t &= \delta_{st}\mathcal{L}_{\frac{n}{2}}u^t \\ F_s &= \mathcal{P}(g^{-1}, \partial^{n-1}g) \end{cases}$$

for $1 \leq s \leq N$.

We now have the following:

Proposition 7.2. *Interpreted as above,* (13) *is uniformly elliptic.*

Proof. For weights we take $w(u^t) = n$ and $w(L_s) = 0$, and so $L'_{st}(\partial) = L_{st}(\partial)$. The principal symbol for $\mathcal{L}$ is $|\xi|_g^2$ and more generally, the principal symbol for $\mathcal{L}_l$ is $|\xi|_g^{2l}$. Hence, for our system, we have

$$\det(L'(\xi)) = |\xi|_g^{nN},$$

from which we may conclude that our system is uniformly elliptic. $\square$

7.2.2 Boundary Equations and the Complementing Condition

Focusing on the boundary equations, our first task is to incorporate (19) into (14) – (18). We cannot simply add (19) to the list of boundary equations since the number of equations we need is determined by the degree of the system. Instead, we will modify the equations of order 1 by differentiating them and substituting certain terms using (19). These new equations together with the rest will be shown to satisfy the complementing condition.

To incorporate (19) into the equations of order 1, we calculate the normal derivative of (15) to get

$$g^{\gamma 0}g^{\eta\beta}g_{\alpha\beta},_{\eta\gamma} - \frac{1}{2}g^{\gamma 0}g^{\eta\beta}g_{\eta\beta},_{\alpha\gamma} = \mathcal{P}(g^{-1}, \partial g).$$

When $\alpha = 0$, this becomes

$$g^{\gamma 0}g^{\eta\beta}g_{0\beta},_{\eta\gamma} - \frac{1}{2}g^{\gamma 0}g^{\eta\beta}g_{\eta\beta},_{0\gamma} = \mathcal{P}(g^{-1}, \partial g). \tag{49}$$

For $\alpha = i$, we can expand some of the sums and rearrange to produce

$$\begin{aligned}(g^{00})^2 g_{0i},_{00} + 2g^{00}g^{0k}g_{0i},_{0k} + g^{0j}g^{0k}g_{0i},_{jk} - g^{0\gamma}g^{0j}g_{0j},_{\gamma i}& \\ + g^{0\gamma}g^{\eta j}g_{ij},_{\eta\gamma} - \frac{1}{2}g^{0\gamma}g^{jk}g_{jk},_{\gamma i} - \frac{1}{2}g^{00}g^{0\gamma}g_{00},_{\gamma i}& \\ = \mathcal{P}(g^{-1}, \partial g)&\end{aligned} \tag{50}$$

We will use (19) to replace the first two terms in this equation. To do this, expand the derivative of the second fundamental form in (19) in terms of the metric and trace to get

$$g^{ij}g_{0i},_{jk} - \frac{1}{2}g^{ij}g_{ij},_{0k} = \frac{(n-1)(g^{00})^{-1}}{2(2-n)}\mathcal{L}g_{0k} + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h).$$

Expanding $\mathcal{L}g_{0k}$, rearranging, multiplying by g^{00}, and changing indices leads to

$$\begin{aligned}(g^{00})^2 g_{0i},_{00} + 2g^{00}g^{0k}g_{0i},_{0k}& \\ = \frac{2(2-n)(g^{00})^2}{n-1}g^{jk}g_{0k},_{ij} - g^{00}g^{jk}g_{0i},_{jk} - \frac{(2-n)(g^{00})^2}{n-1}g^{jk}g_{jk},_{0i}& \\ + \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h).&\end{aligned}$$

Using this to replace the first two terms on the left hand side of (50), we have

$$\begin{aligned}\frac{2(2-n)(g^{00})^2}{n-1}g^{jk}g_{0k},_{ij} - g^{00}g^{jk}g_{0i},_{jk} - \frac{(2-n)(g^{00})^2}{n-1}g^{jk}g_{jk},_{0i} + g^{0j}g^{0k}g_{0i},_{jk}& \\ - g^{0\gamma}g^{0j}g_{0j},_{\gamma i} + g^{0\gamma}g^{\eta j}g_{ij},_{\eta\gamma} - \frac{1}{2}g^{0\gamma}g^{jk}g_{jk},_{\gamma i} - \frac{1}{2}g^{0\gamma}g^{00}g_{00},_{\gamma i}& \\ = \mathcal{P}(g^{-1}, (g^{00})^{-\frac{1}{2}}, \partial g, \partial_t^2 h),&\end{aligned} \tag{51}$$

which, with (49), are used in place of the equations of order 1 to give us the following:

Proposition 7.3. *The equations* (14), (49), (51), (16) – (18) *satisfy the complementing condition for the system* (13).

After introducing some organizing notation and calculating the polynomial M^+ and the matrix adjoint to $L'_{st}(\xi + \tau\nu)$, we prove a technical lemma to help us analyze these equations. Finally, we prove this proposition using an inductive argument.

We have a total of $M = nN/2$ boundary equations so for the components $B_{rt}(\partial)$ we let $1 \leq r \leq M$ and as above, we have $1 \leq t \leq N$. These conditions are grouped by the order of the equations and we will call the set of rows in $B_{rt}(\partial)$ corresponding to the equations of order s the *block of order s*. Additionally, we will refer to the rows corresponding to (16) as the block of order 2(a), while the rows corresponding to (49) and (51) will be denoted the block of order 2(b). As with the system, we relegate all but the highest order parts of each equation to the inhomogeneous term. With $w(u^t) = n$ as indicated above, we let $w(B_r) = s - n$ for any row $B_r(\partial)$ in a block of order s. Hence we have $B'(\partial) = B(\partial)$.

Working out how each of the boundary equations are represented, we find that $B(\partial)$ has the following properties: The leftmost $(N-n)\times(N-n)$ minor in the block of order 0 is the identity matrix I_{N-n}. The remaining entries in the block of order 0 are equal to zero. The leftmost $(N-n)\times(N-n)$ minor in the block of order 2(a) is $\mathcal{L}I_{N-n}$. The remaining entries in the block of order 2(a) are equal to zero. The blocks of order $2l$, $l\geq 2$ are of the form $\mathcal{L}_l I_N$. The rightmost $n\times n$ minor in the block of order 3 is lower triangular and, with the exception of the last row, it is diagonal. The operator on the diagonal is $g^{\eta 0}\partial_\eta\mathcal{L}$, while the operator in the jth entry of the last row is $g^{\eta j}\partial_\eta\mathcal{L}$. We will find that the remainder of this block will not enter into our analysis. Finally, the block of order 2(b) proves to be complicated. Like the block of order 3, we will find that all we need to study is the rightmost $n\times n$ minor. This will be studied more closely later.

We need the polynomial M^+ and the matrix adjoint to $L'_{st}(\xi+\tau\nu)$ in order to check the complementing condition. In our case, we have $\det(L'(\xi+\tau\nu)) = |\xi+\tau\nu|_g^{nN}$. This function has two complex roots, τ^+ and $\tau^- = \overline{\tau^+}$, each with multiplicity $nN/2$. Taking τ^+ to be the root with positive imaginary part we have

$$M^+(\tau) = (\tau-\tau^+)^{\frac{n}{2}N}.$$

Also, calculating the matrix adjoint we have

$$\mathrm{adj}(L')^{tq}(\xi+\tau\nu) = |\xi+\tau\nu|_g^{n(N-1)}\delta^{tq}.$$

Note that, as with the determinant of $L'(\xi+\tau\nu)$, each component here has the two roots τ^+, and τ^-, but this time their multiplicities are each $n(N-1)/2$.

Now, through Lemma 7.1, the complementing condition simplifies in that all of the terms in $\mathrm{adj}L'(\xi+\tau\nu)$ may be reduced to δ^{tq} and in doing so, M^+ is reduced to $(\tau-\tau^+)^{\frac{n}{2}}$. Hence, the only effect that $\mathrm{adj}L'$ has is to raise an index on B' and to keep notation simple, we lower it back down. Therefore, verifying the complementing condition amounts to checking that the rows of $(B')(\xi+\tau\nu)$ are linearly independent modulo $(\tau-\tau^+)^{\frac{n}{2}}$. This follows from a careful analysis of the interaction among the various rows in $B'(\xi+\tau\nu)$. For the most part, this analysis is straightforward. The only real complication is that the rows composing the block of order 2(b) do not have a simple structure. We will find that the rightmost $n\times n$ minor in this block, which we label $\mathcal{F}$, is all we need to analyze. To help in this regard, we have the following lemma:

Lemma 7.4. *In a neighborhood of a point $p\in D$ where $g = id$, the matrix $\mathcal{F}$ is invertible when $\tau=\tau^+$.*

We note that by making a linear change to the coordinates in Proposition 2.1, we can guarantee that the metric is equal to the identity at a point of our choice, so this condition does not have a significant impact on the applicability of these results.

Proof. It is sufficient to analyze $\mathcal{F}$ when $g_{\alpha\beta}=\delta_{\alpha\beta}$. In this case, when $\tau=\tau^+$, the matrix simplifies to

$$\begin{pmatrix} A & v \\ w & b \end{pmatrix},$$

where A is an $(n-1)\times(n-1)$ matrix with $A_{ij} = -\delta_{ij}|\xi|^2 + \frac{2(2-n)}{n-1}\xi_i\xi_j$, v is a column vector with $v_i = -\frac{1}{2}\xi_i\tau^+$, w is a row vector with $w_j = \xi_j\tau^+$, and $b = \frac{1}{2}(\tau^+)^2$.

Performing column operations on this matrix, we get a new matrix

$$\begin{pmatrix} A' & v' \\ w' & b' \end{pmatrix},$$

where $A'_{ij} = -\delta_{ij}|\xi|^2 + \frac{3-n}{n-1}\xi_i\xi_j$, $v'_i = -\frac{1}{2}\xi_i$, $w'_j = 0$ and $b' = \frac{1}{2}\tau^+$.

Since all but the last entry in the bottom row are zero, invertibility of the entire matrix depends on invertibility of A'. Note that $A' = \frac{3-n}{n-1}\xi\xi^T - |\xi|^2 I_{(n-1)}$, so A' is invertible as long as $|\xi|^2$ is not an eigenvalue of $\frac{3-n}{n-1}\xi\xi^T$. It is easy to verify that the eigenvalues of $\frac{3-n}{n-1}\xi\xi^T$ are $\frac{3-n}{n-1}|\xi|^2$ (with multiplicity 1) and 0 (with multiplicity $n-2$). Therefore, A' is invertible, and so the original matrix $\mathcal{F}$ is invertible as well. $\square$

Using this lemma to help deal with the block of order 2(b), we have the following:

Proof of Proposition 7.3. As mentioned above, we need to show that the rows of $B'(\xi+\tau\nu)$ are linearly independent modulo $(\tau-\tau^+)^{\frac{n}{2}}$. That is, given the following:

$$c^r(B')_{rt}(\xi+\tau\nu) = P_t(\tau)(\tau-\tau^+)^{\frac{n}{2}}, \tag{52}$$

where $c^r \in \mathbb{C}$ and P_t are polynomials in one complex variable, then in fact all c^r are zero. Letting C^s be the (row) vector of coefficients c^r corresponding to the rows in the block of order s, we will proceed inductively. First we will show that C^0 and $C^{2(b)}$ must be zero. Then we will show that $C^{2(a)}$ and C^3 must be zero. Finally, we will show that the remaining coefficients are zero.

We note again that $\mathcal{L}(\xi+\tau\nu) = |\xi+\tau\nu|_g^2 = (\tau-\tau^+)(\tau-\tau^-)$. More generally, we have $\mathcal{L}_l(\xi+\tau\nu) = (\tau-\tau^+)^l(\tau-\tau^-)^l$. Also, the operators in the rightmost $n\times n$ minor in the block of order 3 each include an $\mathcal{L}$ and a sum of first order operators. Hence each of these terms take the form $g^{\alpha\eta}(\tau-\tau^+)(\tau-\tau^-)(\xi_\eta+\tau\nu_\eta)$ which has one real root since it is cubic and all its coefficients are real. We can then write this minor as $(\tau-\tau^+)(\tau-\tau^-)\mathcal{T}$ where $\mathcal{T}$ is a lower triangular $n\times n$ matrix with $g^{0\eta}(\xi_\eta+\tau\nu_\eta)$ for entries on the diagonal and $g^{i\eta}(\xi_\eta+\tau\nu_\eta)$ for the ith entry of the last row. With this we can write down (52) as follows:

$$\begin{aligned}
&(C^0\ C^{2(b)})\begin{pmatrix} I_{N-n} & 0 \\ * & \mathcal{F}\end{pmatrix} + (\tau-\tau^+)(\tau-\tau^-)(C^{2(a)}\ C^3)\begin{pmatrix} I_{N-n} & 0 \\ * & \mathcal{T}\end{pmatrix} \\
&+C^4(\tau-\tau^+)^2(\tau-\tau^-)^2 I_N + \ldots + C^{n-2}(\tau-\tau^+)^{\frac{n}{2}-1}(\tau-\tau^-)^{\frac{n}{2}-1} I_N \\
&\qquad\qquad = \ (P_1(\tau)\ \ldots\ P_N(\tau))(\tau-\tau^+)^{\frac{n}{2}}
\end{aligned} \tag{53}$$

For the first step, (53) must be true in particular at $\tau=\tau^+$. With the exception of the first term, every term is 0 at $\tau=\tau^+$. Since the matrix in the first term is invertible at τ^+ (with the help of Lemma 7.4), C^0 and $C^{2(b)}$ must be zero.

The analysis for the next term in (53) is similar. We divide by $(\tau-\tau^+)$ and since the equation must still be true at $\tau=\tau^+$, we may conclude that $C^{2(a)}$ and C^3 must be zero. Continuing by dividing by $(\tau-\tau^+)$ and then evaluating at $\tau=\tau^+$, we see that all the remaining C^s must be zero as well.

Therefore all c^r must be zero and the rows of $B'(\xi+\tau\nu)$ are linearly independent modulo $(\tau-\tau^+)^{\frac{n}{2}}$. $\square$

7.3 Regularity

Once we know that our system satisfies the complementing condition, we can apply the boundary regularity results of [ADN64] or [Mor66] with the goal of proving Theorems A and B. Unfortunately it seems that these regularity results do not give us quite the results we want. In particular, the results in [ADN64] are local, but, in our context, they also require that $g \in C^{n,\sigma}(U)$. On the other hand, there are weak regularity results in [Mor66], but they are global on a domain in $\mathbb{R}^n$ and require strong regularity transverse to the boundary. The global nature makes it unclear how to account for the use of harmonic coordinates in this setting. Local/global issues aside, in order to satisfy the regularity conditions at the boundary, g is required to be in $C^{4,\sigma}(\overline{M})$ in dimension 4 and $C^{n-1,\sigma}(\overline{M})$ in higher dimensions, the difference arising because of the fact that in dimension 4, there is boundary data up to order 3, while in dimension n greater than 4, there is boundary data only up to order $n-2$. Therefore, neither approach improves the bootstrap approach presented here.

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